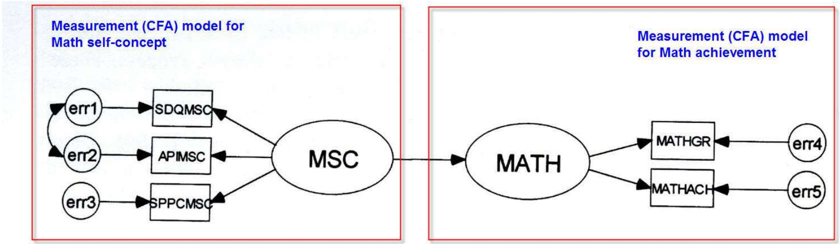
Exploratory and Confirmatory Factor Analysis

Part 3: CFA & SEM models

Michael Friendly

Psychology 6140



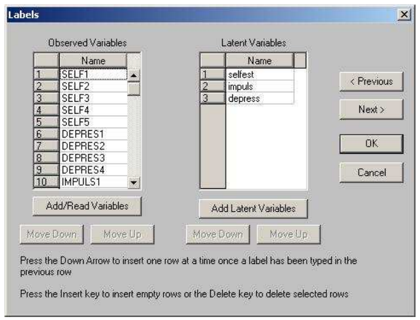
Part 3: CFA Outline

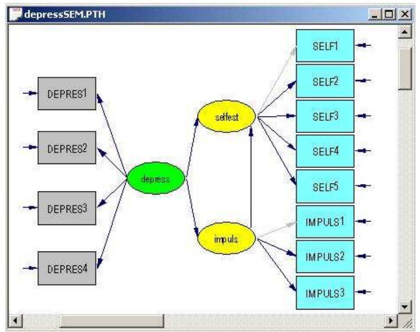
Confirmatory Factor Analysis

- Preludes: CFA software, path diagrams, caveats
- Development of CFA models
  - Restricted maximum likelihood factor analysis (RIMLFA model)
  - Analysis of covariance structures (ACOVS model)
  - Structural equations, mean structures (LISREL model)
- Applications of CFA
  - Higher-order factor analysis
  - Test theory models of "equivalence"
  - Sets of congeneric tests
  - Inter-rater reliability
  - Multi-trait, Multi-method data
  - Simplex models for ordered latent variables
  - Factorial invariance
  - Power and sample size for EFA and CFA models

Prelude: CFA software

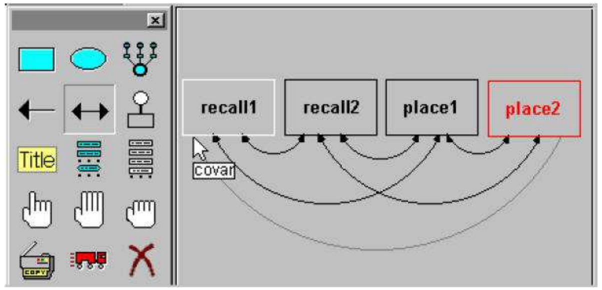
- LISREL (http://www.ssicentral.com/)
  - Originally designed as stand-alone program with matrix syntax
  - LISREL 8.5+ for Windows/Mac: Includes
    - interactive, menu-driven version;
    - PRELIS (pre-processing, correlations and models for categorical variables);
    - SIMPLIS (simplified, linear equation syntax)
    - path diagrams from the fitted model





Prelude: CFA software

- Amos (http://www.spss.com/amos/): Linear equation syntax + path diagram model description
  - import data from SPSS, Excel, etc; works well with SPSS
  - Create the model by drawing a path diagram
  - simple facilities for multi-sample analyses
  - nice comparative displays of multiple models



Prelude: CFA software

- SAS 9.3: PROC CALIS
  - MATRIX (à la LISREL), LINEQS (à la EQS), RAM, ... syntax
  - Now handles multi-sample analyses
  - Multiple-model analysis syntax, e.g., Model 2 is like Model 1 except ...
  - Enhanced output controls
  - customizable fit summary table
- SAS macros `http://datavis.ca/sasmac/`:
  - `caliscmp` macro: compare model fits from PROC CALIS à la Amos
  - `csmpower` macro: power estimation for covariance structure models

R: sem, lavaan and others

- `sem` package package (John Fox)
  - flexible ways to specify models: `cfa()`, `linearEquations()`, and `multigroupModel()`
  - `bootSem()` provides bootstrap analysis of SEM models
  - `miSem()` provides multiple imputation
  - path diagrams using `pathDiagram()` → `graphviz`
  - `polychor` package for polychoric correlations
- `lavaan` package package (Yves Rossell)
  - Functions `lavaan()`, `cfa()`, `sem()`, `growth()` (growth curve models)
  - Handles multiple groups models
  - `semTools` package provides tests of measurement invariance, multiple imputation, bootstrap analysis, power analysis for RMSEA, ...
- `semPlot` package package — path diagrams for `sem` package, `lavaan` package, Mplus, ... models

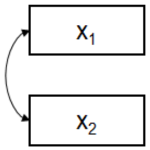
Mplus

Mplus `https://www.statmodel.com/` [$$$, but cheaper student price]

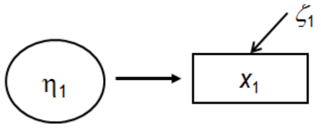
- Handles the widest range of models: CFA, SEM, multi-group, multi-level, latent group
- Variables: continuous, censored, binary, ordered categorical (ordinal), unordered categorical (nominal), counts, or combinations of these variable types
- For binary and categorical outcomes: probit, logistic regression, or multinomial logistic regression models.
- For count outcomes: Poisson and negative binomial regression models.
- Extensive facilities for simulation studies.

Prelude: Path diagrams

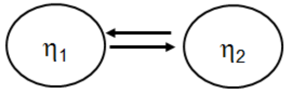
- Visual representation of a set of simultaneous equations for EFA, CFA, SEM models



- Rectangles and square boxes represent observed (manifest) variables
- Double-headed curved arrows represent unanalyzed association (correlation)



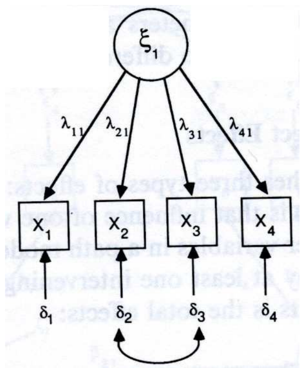
- Ellipses and circles represent unobserved (latent) variables
- Straight, single-headed arrows indicate causal relations from base to head
- Unenclosed symbol is an error term (in equation or measurement error)



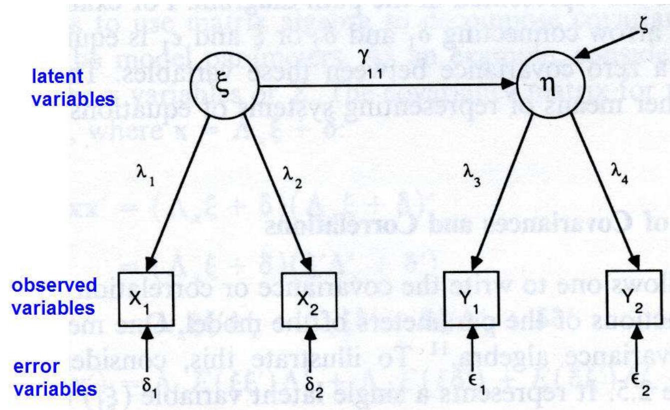
- Double single-headed arrows indicate reciprocal causation
- Allows different weights for each path

Prelude: Path diagrams

Schematic Examples:



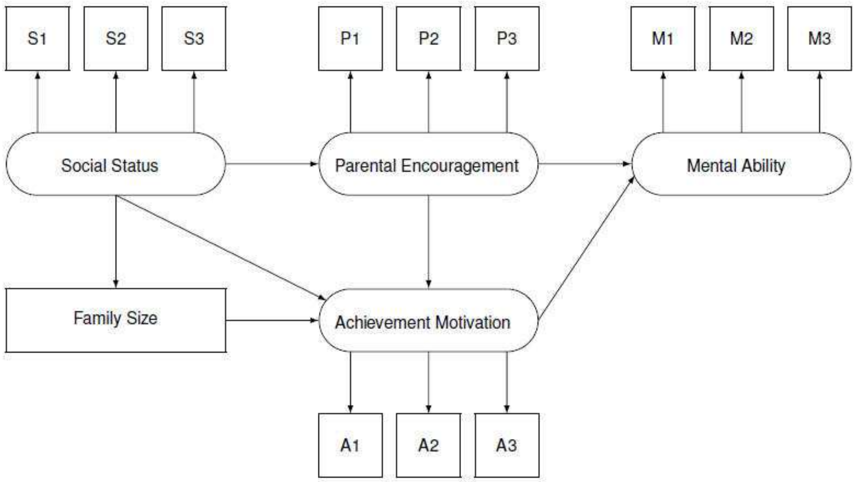
CFA, 1-factor model (correlated errors)



SEM, two latent variables, each with two indicators

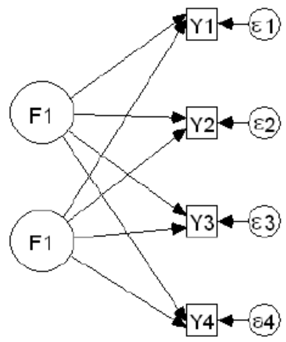
Prelude: Path diagrams

Substantive Example: SEM with multiple indicators, path model for latent variables

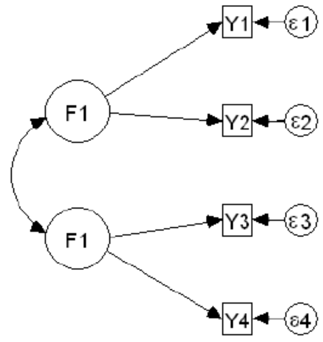


EFA vs. CFA

Exploratory FA	Confirmatory FA
# of factors is unknown	# of factors are set from theory
loading pattern is unknown	loading pattern is determined from theory
(e.g. all factors load on all variables)	(e.g. $Y_1 = l_1 F_1 + \epsilon$)
can be unidentifiable	usually identifiable
factors are uncorrelated	factors can be correlated



Exploratory FA



Confirmatory FA

Prelude: Caveats

- CFA and SEM models are fit using the covariance matrix (***S***)
  - The raw data is often not analyzed
- Typically, this assumes all variables are complete, continuous, multivariate normal. Implies:
  - ***S*** is a sufficient statistical summary
  - Goodness-of-fit (χ^2) and other tests based on asymptotic theory ($N \to \infty$)
  - Missing data, skewed or long-tailed variables must be handled first
- Topics not covered here:
  - Using polychoric correlations for categorical indicators
  - Distribution-free estimation methods (still asymptotic)
  - Bootstrap methods to correct for some of the above

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Outline

Indeterminacy of the Common Factor Model

- The general Factor Analysis model allows the factors to be correlated. Let $\mathbf{\Phi}_{k\times k}$ be the variance-covariance matrix of the common factors. Then the model is

$$\mathbf{\Sigma} = \mathbf{\Lambda\Phi\Lambda}^{\mathsf{T}} + \mathbf{\Psi} \tag{6}$$

- However, model (1) is not ***identified***, since it has more parameters than there are correlations (or variances and covariances).
- That is, any rotation of $\mathbf{\Lambda}$ by a non-singular transformation matrix, $\boldsymbol{T}_{k\times k}$ will fit equally well:

$$F(\mathbf{\Lambda}, \mathbf{\Phi}) = F(\mathbf{\Lambda}\boldsymbol{T},\ \boldsymbol{T}^{-1}\mathbf{\Phi}\boldsymbol{T}^{-1})$$

- The transformation matrix, $\boldsymbol{T}_{(k\times k)}$ corresponds to the fact that k^2 restrictions need to be imposed in $\mathbf{\Phi}$ and/or $\mathbf{\Lambda}$ to obtain a unique solution.
- Setting $\mathbf{\Phi} = I$ (uncorrelated factors) gives $k(k+1)/2$ restrictions; all methods of estimating factor impose an additional $k(k-1)/2$ restrictions on $\mathbf{\Lambda}$.

Indeterminacy of the Common Factor Model

- Therefore, the number of effective unknown parameters is:

$$\overbrace{pk}^{\mathbf{\Lambda}} + \overbrace{k(k+1)/2}^{\mathbf{\Phi}} + \overbrace{p}^{\mathbf{\Psi}} - \overbrace{k^2}^{\boldsymbol{T}} = pk + p - k(k-1)/2$$

so the number of ***degrees of freedom*** for the model is:

Sample moments ($\boldsymbol{S}$)	$p(p+1)/2$
- Parameters estimated	$pk + p - k(k-1)/2$
= Degrees of freedom	$[(p-k)^2 - (p+k)]/2$

- E.g., with $p = 6$ tests, $k = 3$ factors will always fit perfectly

k	1	2	3
df	9	4	$\boxed{0}$

Restricted Maximum Likelihood FA

The essential ideas of CFA can be introduced most simply as follows:

- Jöreskog (1969) proposed that a factor hypothesis could be tested by imposing restrictions on the factor model, in the form of **fixed** elements in Λ and Φ (usually 0).
- The maximum likelihood solution is then found for the remaining **free** parameters in Λ and Φ.
- The χ^2 for the restricted solution provides a test of how well the hypothesized factor structure fits.

Restricted Maximum Likelihood FA

For example, the pattern below specifies two non-overlapping oblique factors, where the x's are the only free parameters.

$$\Lambda = \begin{bmatrix} x & 0 \\ x & 0 \\ x & 0 \\ 0 & x \\ 0 & x \\ 0 & x \end{bmatrix} \qquad \Phi = \begin{bmatrix} 1 & \\ x & 1 \end{bmatrix}$$

- This CFA model has only 7 free parameters and $df = 15 - 7 = 8$.
- A $k = 2$-factor EFA model would have all parameters free and $df = 15 - 11 = 4$ degrees of freedom.
- If this restricted model fits (has a small χ^2/df), it is strong evidence for two non-overlapping oblique factors.
- This is a more precise hypothesis than can be tested by EFA + rotation.

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Restricted vs. Unrestricted solutions

Unrestricted solution

Factor solutions with $m = k^2$ restrictions are mathematically equivalent:

- same communalities,
- same goodness of fit χ^2.
- Any unrestricted solution can be rotated to any other.

Restricted solution

Solutions with $m > k^2$ restrictions

- have different communalities,
- do not reflect the same common factor space, and
- cannot be rotated to one another.

All true CFA models call for restricted solutions.

Example: Ability and Aspiration

Calsyn & Kenny (1971) studied the relation of perceived ability and educational aspiration in 556 white eigth-grade students. Their measures were:

x_1: self-concept of ability
x_2: perceived parental evaluation
x_3: perceived teacher evaluation
x_4: perceived friend's evaluation
x_5: educational aspiration
x_6: college plans

- Their interest was primarily in estimating the correlation between "true (perceived) ability" and "true apsiration".
- There is also interest in determining which is the most reliable indicator of each latent variable.

The correlation matrix is shown below:

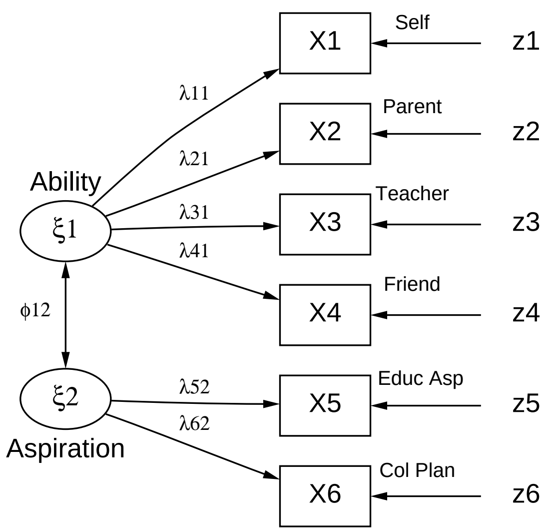
	S-C	Par	Tch	Frnd	Educ	Col
S-C Abil	1.00					
Par Eval	0.73	1.00				
Tch Eval	0.70	0.68	1.00			
FrndEval	0.58	0.61	0.57	1.00		
Educ Asp	0.46	0.43	0.40	0.37	1.00	
Col Plan	0.56	0.52	0.48	0.41	0.72	1.00
	x1	x2	x3	x4	x5	x6

The model to be tested is that

- x_1-x_4 measure only the latent "ability" factor and
- x_5-x_6 measure only the "aspiration" factor.
- If so, are the two factors correlated?

Specifying the model

The model can be shown as a path diagram:



Specifying the model

This can be cast as the restricted factor analysis model:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & 0 \\ \lambda_{31} & 0 \\ \lambda_{41} & 0 \\ 0 & \lambda_{52} \\ 0 & \lambda_{62} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}$$

If this model fits, the questions of interest can be answered in terms of the estimated parameters of the model:

- Correlation of latent variables: The estimated value of $\phi_{12} = r(\xi_1, \xi_2)$.
- Reliabilities of indicators: The communality, e.g., $h_i^2 = \lambda_{i1}^2$ is the estimated reliability of each measure.

The solution (found with LISREL and `PROC CALIS`) has an acceptable fit:

$$\chi^2 = 9.26 \quad \text{df} = 8 \quad (p = 0.321)$$

The estimated parameters are:

	LAMBDA X Ability	Aspiratn	Communality	Uniqueness
S-C Abil	0.863	0	0.745	0.255
Par Eval	0.849	0	0.721	0.279
Tch Eval	0.805	0	0.648	0.352
FrndEval	0.695	0	0.483	0.517
Educ Asp	0	0.775	0.601	0.399
Col Plan	0	0.929	0.863	0.137

Thus,

- Self-Concept of Ability is the most reliable measure of ξ_1, and College Plans is the most reliable measure of ξ_2.
- The correlation between the latent variables is $\phi_{12} = .67$. Note that this is higher than any of the individual between-set correlations.

Using PROC CALIS

```
data calken(TYPE=CORR);
   _TYPE_ = 'CORR'; input _NAME_ $ V1-V6;      % $
   label V1='Self-concept of ability'
         V2='Perceived parental evaluation'
         V3='Perceived teacher evaluation'
         V4='Perceived friends evaluation'
         V5='Educational aspiration'
         V6='College plans';
   datalines;
V1     1.      .       .       .       .       .
V2      .73    1.      .       .       .       .
V3      .70     .68    1.      .       .       .
V4      .58     .61     .57    1.      .       .
V5      .46     .43     .40     .37    1.      .
V6      .56     .52     .48     .41     .72    1.
  ;
```

Using PROC CALIS

The CFA model can be specified in several ways:

- With the FACTOR statement, specify names for the free parameters in $\boldsymbol{\Lambda}$ (MATRIX _F_) and $\boldsymbol{\Phi}$(MATRIX _P_)

```
proc calis data=calken method=max edf=555 short mod;
 FACTOR n=2;
 MATRIX _F_                          /* loadings */
          [ ,1] = lam1-lam4 ,        /* factor 1 */
          [ ,2] = 4 * 0 lam5 lam6 ;  /* factor 2 */
 MATRIX _P_
          [1,1] = 2 * 1. ,
          [1,2] = COR;      /* factor correlation */
 run;
```


Using PROC CALIS

- With the LINEQS statement, specify linear equations for the observed variables, using F1, F2, ... for common factors and E1, E2, ... for unique factors.
- STD statement specifies variances of the factors
- COV statement specifies covariances

```
proc calis data=calken method=max edf=555;
 LINEQS
      V1 = lam1 F1           + E1 ,
      V2 = lam2 F1           + E2 ,
      V3 = lam3 F1           + E3 ,
      V4 = lam4 F1           + E4 ,
      V5 =         lam5 F2 + E5 ,
      V6 =         lam6 F2 + E6 ;
 STD
      E1-E6 = EPS: ,
      F1-F2 = 2 * 1. ;
 COV
      F1 F2 = COR ;
 run;
```


Using cfa() in the sem package

In v. 2 of the sem package, CFA models are really easy to specify using the cfa() function.

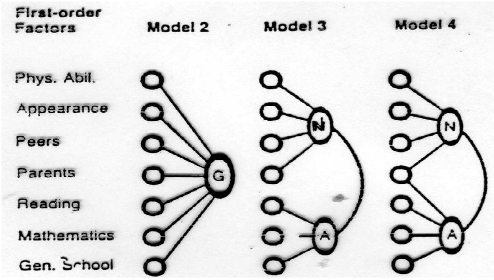
```
library(sem)
mod.calken <- cfa()
   F1: v1, v2, v3, v4
   F2: v5, v6

fit.calken <- sem(mod.calken, R.calken, N=556)
```

- Options allow you to specify reference indicators, and to specify covariances among the factors, allowing the factors to be correlated or uncorrelated.
- By default, all factors in CFA models are allowed to be correlated, simplifying model specification.
- The sem package now includes edit() and update() functions, allowing you to delete, add, replace, fix, or free a path or parameter in a semmod object.

Outline

Higher-order factor analysis

- In EFA & CFA, we often have a model that allows the factors to be correlated ($\Phi \neq I$)
- If there are more than a few factors, it sometimes makes sense to consider a 2nd-order model, that describes the correlations among the 1st-order factors.
- In EFA, this was done simply by doing another factor analysis of the estimated factor correlations $\widehat{\Phi}$ from the 1st-order analysis (after an oblique rotation)
- The second stage of development of CFA models was to combine these steps into a single model, and allow different hypotheses to be compared.



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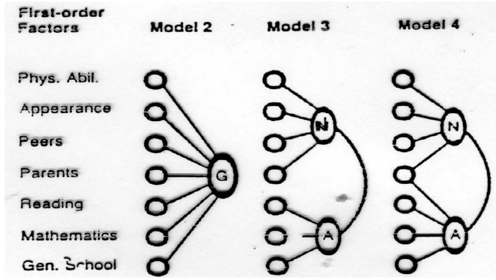

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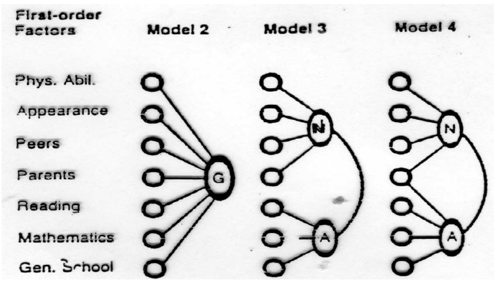

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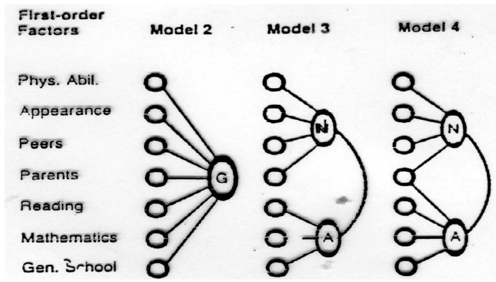


Analysis of Covariance Structures (ACOVS)

Jöreskog (1970, 1974) proposed a generalization of the common factor model which allows for second-order factors.

$$\begin{aligned} \boldsymbol{\Sigma} &= \boldsymbol{B}(\boldsymbol{\Lambda}\boldsymbol{\Phi}\boldsymbol{\Lambda}^{\mathsf{T}} + \boldsymbol{\Psi}^2)\boldsymbol{B}^{\mathsf{T}} + \boldsymbol{\Theta}^2 \\ &= \boldsymbol{B}\boldsymbol{\Gamma}\boldsymbol{B}^{\mathsf{T}} + \boldsymbol{\Theta}^2 \end{aligned}$$

where:

- $\boldsymbol{B}_{(p\times k)}$ = loadings of observed variables on k 1st-order factors.
- $\boldsymbol{\Gamma}_{(k\times k)}$ = correlations among 1st-order factors.
- $\boldsymbol{\Theta}^2_{(p\times p)}$ = diagonal matrix of unique variances of 1st-order factors.
- $\boldsymbol{\Lambda}_{(k\times r)}$ = loadings of 1st-order factors on r second-order factors.
- $\boldsymbol{\Phi}_{(r\times r)}$ = correlations among 2nd-order factors.
- $\boldsymbol{\Psi}^2$ = diagonal matrix of unique variances of 2nd-order factors.

The model is thus a nesting of a 2nd-order model for $\boldsymbol{\Gamma}$ within the 1st-order model for $\boldsymbol{\Sigma}$.

Analysis of Covariance Structures (ACOVS)

In applications of ACOVS, any parameters in $\boldsymbol{B}, \boldsymbol{\Lambda}, \boldsymbol{\Phi}, \boldsymbol{\Psi}$, or $\boldsymbol{\Theta}$ may be

- **free** to be estimated,
- **fixed constants** by hypothesis, or
- **constrained** to be equal to other parameters.

The maximum likelihood solution minimizes:

$$F(\boldsymbol{B}, \boldsymbol{\Lambda}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}) = \mathrm{tr}(S\widehat{\boldsymbol{\Sigma}}^{-1}) + \log|\widehat{\boldsymbol{\Sigma}}| - \log|S| - p$$

with respect to the independent free parameters. At the minimum, $(N-1)F_{min} \sim \chi^2$ with degrees of freedom $= p(p+1)/2$ - (number of free parameters in model).

Example: 2nd Order Analysis of Self-Concept Scales

A theoretical model of self-concept by Shavelson & Bolus (1976) describes facets of an individual's self-concept and presents a hierarchical model of how those facets are arranged.
To test this theory, Marsh & Hocevar (1985) analyzed measures of self-concept obtained from 251 fifth grade children with a Self-Description Questionnaire (SDQ). 28 subscales (consisting of two items each) of the SDQ were determined to tap four non-academic and three academic facets of self-concept:

- physical ability
- physical appearance
- relations with peers
- relations with parents
- reading
- mathematics
- general school

Example: 2nd Order Analysis of Self-Concept Scales

The subscales of the SDQ were determined by a first-order exploratory factor analysis. A second-order analysis was carried out examining the correlations among the first-order factors to examine predictions from the Shavelson model(s).

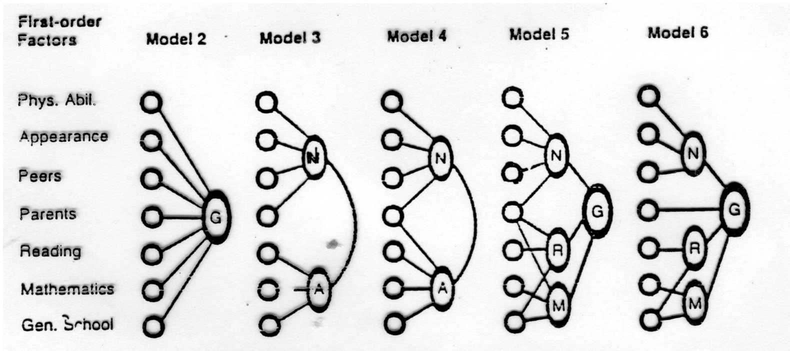


Figure 1. Higher order factor structures for Models 2 to 6.

sem package: Second-order CFA, Thurstone data

Data

Data on 9 ability variables:

```
R.thur <- readMoments(diag=FALSE, names=c('Sentences',
  'Vocabulary', 'Sent.Completion','First.Letters',
  '4.Letter.Words','Suffixes','Letter.Series','Pedigrees',
  'Letter.Group'))
    .828
    .776    .779
    .439    .493     .46
    .432    .464     .425    .674
    .447    .489     .443    .59     .541
    .447    .432     .401    .381    .402    .288
    .541    .537     .534    .35     .367    .32    .555
    .38     .358     .359    .424    .446    .325   .598    .452
```


sem package: Second-order CFA, Thurstone data

Using the `specifyEquations()` syntax:

```
mod.thur.eq <- specifyEquations()
   Sentences       = lam11*F1
   Vocabulary      = lam21*F1
   Sent.Completion = lam31*F1
   First.Letters   =          lam42*F2
   4.Letter.Words  =          lam52*F2
   Suffixes        =          lam62*F2
   Letter.Series   =                   lam73*F3
   Pedigrees       =                   lam83*F3
   Letter.Group    =                   lam93*F3
   F1 = gam1*F4           # factor correlations
   F2 = gam2*F4
   F3 = gam3*F4
   V(F1) = 1              # factor variances
   V(F2) = 1
   V(F3) = 1
   V(F4) = 1
```

Each line gives a regression equation or the specification of a factor variance (V) or covariance (C)


```
(fit.thur <- sem(mod.thur.eq, R.thur, 213))
 Model Chisquare =  38.2   Df =  24

 lam11  lam21  lam31  lam41  lam52  lam62  lam73  lam83  lam93   g
0.5151 0.5203 0.4874 0.5211 0.4971 0.4381 0.4524 0.4173 0.4076 1.4
  gam2   gam3    th1    th2    th3    th4    th5    th6    th7
1.2538 1.4066 0.1815 0.1649 0.2671 0.3015 0.3645 0.5064 0.3903 0.4
   th9
0.5051
```

More detailed output is provided by `summary()`:

```
summary(sem.thur)
 Model Chisquare =  38.196   Df =  24 Pr(>Chisq) = 0.033101
 Chisquare (null model) =  1101.9   Df =  36
 Goodness-of-fit index =  0.95957
 Adjusted goodness-of-fit index =  0.9242
 RMSEA index =  0.052822   90% CI: (0.015262, 0.083067)
 Bentler-Bonnett NFI =  0.96534
 Tucker-Lewis NNFI =  0.98002
 Bentler CFI =  0.98668
 SRMR =  0.043595
 BIC =  -90.475
 ...
```

sem package: Second-order CFA, Thurstone data

- The same model can be specified using `cfa()`, designed specially for confirmatory factor models
- Each line lists the variables that load on a given factor.

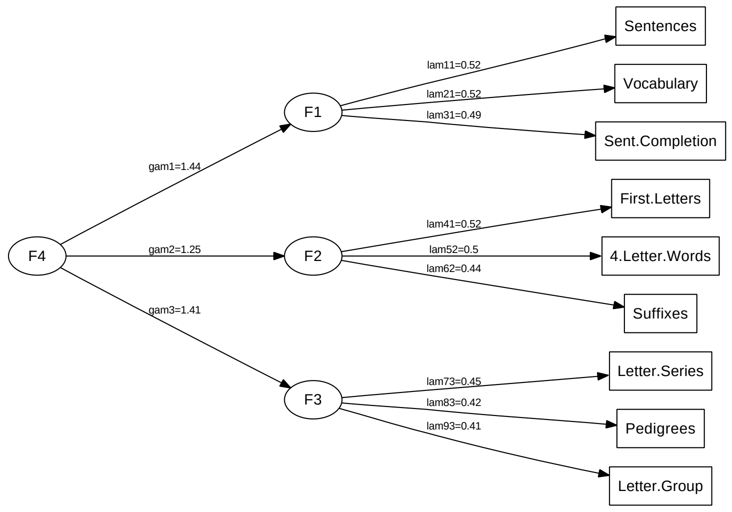
```
mod.thur.cfa <- cfa(reference.indicators=FALSE,
                    covs=c("F1", "F2", "F3", "F4"))
   F1: Sentences, Vocabulary, Sent.Completion
   F2: First.Letters, 4.Letter.Words, Suffixes
   F3: Letter.Series, Pedigrees, Letter.Group
   F4: F1, F2, F3

sem.thur.cfa <- sem(mod.thur.cfa, R.thur, 213)
```

Path diagram:

```
pathDiagram(sem.thur, file="sem-thurstone", edge.labels="both")
```

```
Running  dot -Tpdf -o sem-thurstone.pdf  sem-thurstone.dot
```



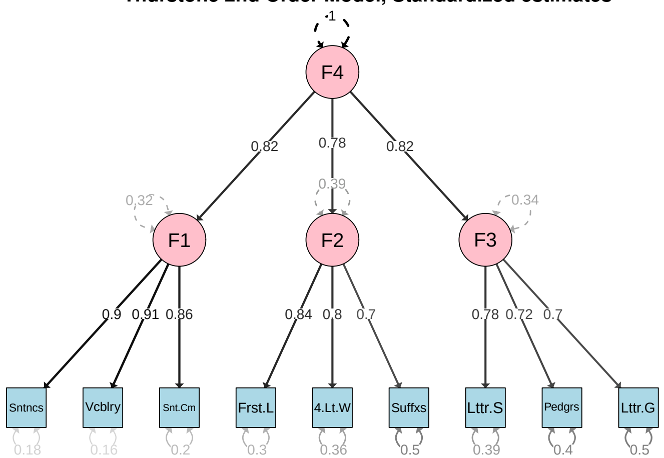
sem package: Other features

- With raw data input, `sem` provides `robust` estimates of standard errors and robust tests
- Can accommodate missing data, via full-information maximum likelihood (FIML)
- `miSem()` generates multiple imputations of missing data using the `mi` package
- `bootSem()` provides nonparametric bootstrap estimates by independent random sampling
- A given model can be easily modified via `edit()` and `update()` methods
- Multiple-group analyses and tests of factorial invariance: `multigroupModel()`.
- Related: `semPlot` package: lovely, flexible, pub. quality path diagrams

Path diagram from `semPlot` package

```
library(semPlot)
semPaths(sem.thur, what="std", color=list(man="lightblue", lat="pink"),
        nCharNodes=6, sizeMan=6, edge.color="black")
title("Thurstone 2nd Order Model, Standardized estimates", cex=1.5)
```

Thurstone 2nd Order Model, Standardized estimates



Outline

LISREL/SEM Model

- Jöreskog (1973) further generalized the ACOVS model to include structural equation models along with CFA.
- Two parts:

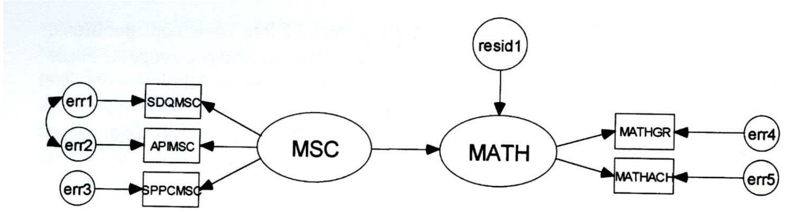
  Measurement model - How the latent variables are measured in terms of the observed variables; measurement properties (reliability, validity) of observed variables. [Traditional factor analysis models]

  Structural equation model - Specifies causal relations among observed and latent variables.
  - Endogenous variables - determined within the model (ys)
  - Exogenous variables - determined outside the model (xs)

Structural eqn. for latent variables	$\eta = \boldsymbol{B}\eta + \Gamma\xi + \zeta$
Measurement models for observed variables	$\boldsymbol{x} = \Lambda_x \xi + \delta$ $\boldsymbol{y} = \Lambda_y \eta + \epsilon$

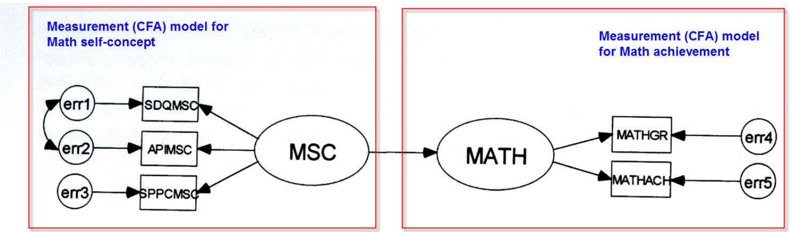
LISREL/SEM Model

SEM model for measures of Math Self-Concept and MATH achievement:



LISREL/SEM Model

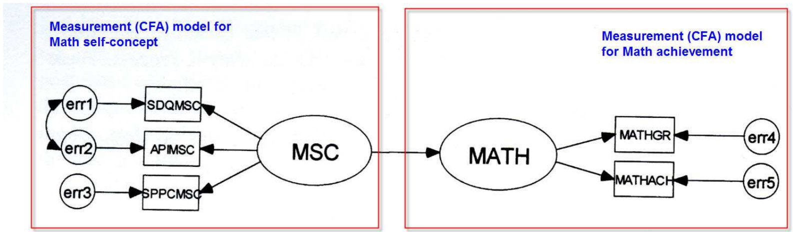
Measurement sub-models for $\boldsymbol{x}$ and $\boldsymbol{y}$



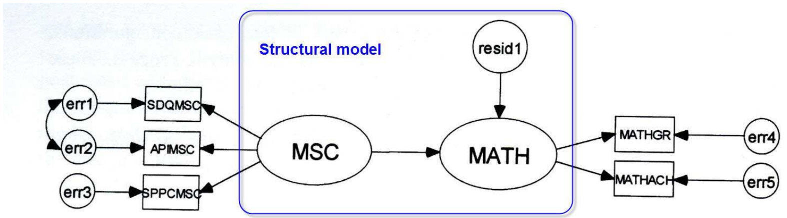
Structural model, relating ξ to η

LISREL/SEM Model

Measurement sub-models for $\boldsymbol{x}$ and $\boldsymbol{y}$



Structural model, relating ξ to η



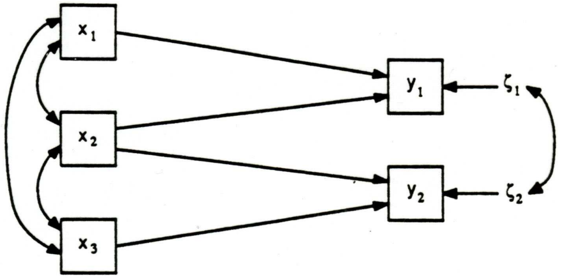
LISREL submodels

- NX = # of observed eXdogenous x variables
- NY = # of observed endogenous y variables
- NKsi = # of latent eXdogenous ξ variables
- NEta = # of latent endogenous η variables
- ***Structural equations, GLM*** $[NY > 0, NX > 0, NK = NE = 0]$

$$\boldsymbol{y} = \boldsymbol{B}\boldsymbol{y} + \boldsymbol{\Gamma}\boldsymbol{x} + \boldsymbol{\zeta}$$

Path analysis, structural equation causal model for directly observed variables. Ordinary regression models and GLM are the special case:

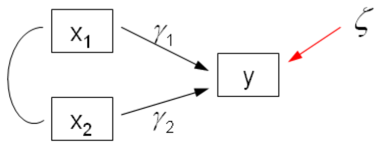
$$\boldsymbol{B} = 0 \Rightarrow \boldsymbol{y} = \boldsymbol{\Gamma}\boldsymbol{x} + \boldsymbol{\zeta}$$



Structural equations: GLM examples

Multiple regression

$$y_i = \gamma_1 x_{1i} + \gamma_2 x_{2i} + \zeta_i$$



Multivariate multiple regression

$$\begin{cases} y_{1i} = \gamma_{11} x_{1i} + \gamma_{12} x_{2i} + \zeta_{1i} \\ y_{2i} = \gamma_{21} x_{1i} + \gamma_{22} x_{2i} + \zeta_{2i} \end{cases}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

$$\mathbf{y} = \mathbf{\Gamma x} + \boldsymbol{\zeta}$$

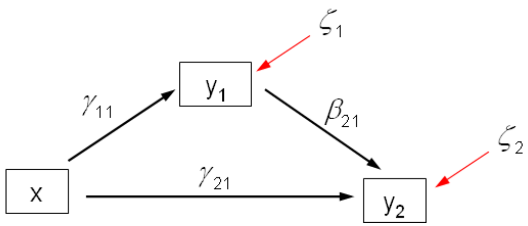
Structural equations: Path analysis example

Simple mediation model: y1 as mediator of (x, y2) relation

$$\begin{cases} y_{1i} = \gamma_{11} x_i + \zeta_{1i} \\ y_{2i} = \gamma_{21} x_i + \beta_{21} y_{1i} + \zeta_{2i} \end{cases}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ \beta_{21} & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \gamma_{11} \\ \gamma_{21} \end{pmatrix} x + \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

$$\mathbf{y} = \mathbf{By} + \mathbf{\Gamma x} + \boldsymbol{\zeta}$$



Measurement models

- ***Factor analysis model*** $[NX > 0, NK > 0, NY = NE = 0]$

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{\Lambda}_x \boldsymbol{\xi} + \boldsymbol{\delta} \\ \Rightarrow \boldsymbol{\Sigma}_{xx} &= \boldsymbol{\Lambda}_x \boldsymbol{\Phi} \boldsymbol{\Lambda}_x^{\mathsf{T}} + \boldsymbol{\Theta}_\delta \end{aligned}$$

- ***Second-order factor analysis model*** $[NY, NE, NK > 0]$

$$\begin{aligned} \boldsymbol{\eta} &= \boldsymbol{\Gamma}\boldsymbol{\xi} + \boldsymbol{\zeta} \\ \boldsymbol{y} &= \boldsymbol{\Lambda}_y \boldsymbol{\eta} + \boldsymbol{\epsilon} \\ \Rightarrow \boldsymbol{\Sigma}_{yy} &= \boldsymbol{\Lambda}_y (\boldsymbol{\Gamma}\boldsymbol{\Phi}\boldsymbol{\Gamma}^{\mathsf{T}} + \boldsymbol{\Theta}_\delta) \boldsymbol{\Lambda}_y^{\mathsf{T}} + \boldsymbol{\Theta}_\epsilon \end{aligned}$$

Test theory is concerned with ideas of reliability, validity and equivalence of measures.

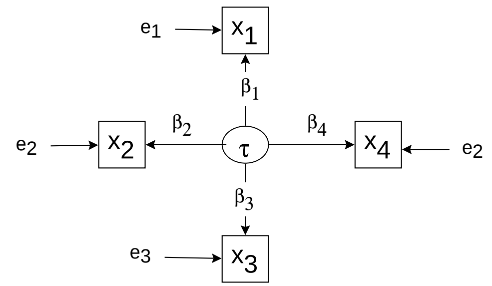
- The same ideas apply to other constructs (e.g., anxiety scales or experimental measures of conservation).
- Test theory defines several degrees of "equivalence".
- Each kind may be specified as a confirmatory factor model with a single common factor.

$$\boldsymbol{\Sigma} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} [\ \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4\ ] + \begin{bmatrix} \theta_1^2 & & & \\ & \theta_2^2 & & \\ & & \theta_3^2 & \\ & & & \theta_4^2 \end{bmatrix}$$

One-factor model:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} [\ \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4\ ] + \begin{bmatrix} \theta_1^2 & & & \\ & \theta_2^2 & & \\ & & \theta_3^2 & \\ & & & \theta_4^2 \end{bmatrix}$$

Path diagram:



Kinds of equivalence

- ***Parallel tests***: Measure the same thing with equal precision. The strongest form of "equivalence".
- ***Tau-equivalent tests***: Have equal true score variances (β_i^2), but may differ in error variance (θ_i^2). Like parallel tests, this requires tests of the same length & time limits. E.g., short forms cannot be τ-equivalent.
- ***Congeneric tests***: The weakest form of equivalence: All tests measure a single common factor, but the loadings & error variances may vary.

These hypotheses may be tested with ACOVS/LISREL by testing equality of the factor loadings (β_i) and unique variances (θ_i^2).

$$\underbrace{\overbrace{\beta_1 = \beta_2 = \beta_3 = \beta_4}^{\tau \text{ equivalent}} \qquad \theta_1^2 = \theta_2^2 = \theta_3^2 = \theta_4^2}_{\text{Parallel}}$$

For several sets of measures, the test theory ideas of congeneric tests can be extended to test the equivalence of each set.
If each set is congeneric, the estimated correlations among the latent factors measure the strength of relations among the underlying "true scores".

Example: Correcting for Unreliability

- Given two measures, x and y, the correlation between them is limited by the reliability of each.
- CFA can be used to estimate the correlation between the true scores, τ_x, τ_y, or to test the hypothesis that the true scores are perfectly correlated:

$$H_0 : \rho(\tau_x, \tau_y) = 1$$

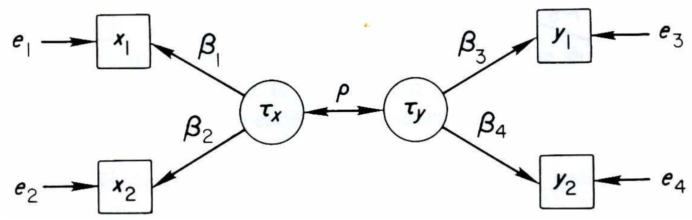
- The estimated true-score correlation, $\hat{\rho}(\tau_x, \tau_y)$ is called the "correlation of x, y corrected for attenuation."

Several Sets of Congeneric Tests

The analysis requires two parallel forms of each test, x_1, x_2, y_1, y_2. Tests are carried out with the model:

$$\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \beta_1 & 0 \\ \beta_2 & 0 \\ 0 & \beta_3 \\ 0 & \beta_4 \end{bmatrix} \begin{bmatrix} \tau_x \\ \tau_y \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \boldsymbol{\Lambda \tau} + \boldsymbol{e}$$

with $corr(\boldsymbol{\tau}) = \rho$, and $var(\boldsymbol{e}) = \mathrm{diag}(\theta_1^2, \theta_2^2, \theta_3^2, \theta_4^2)$. The model is shown in this path diagram:



Several Sets of Congeneric Tests

Hypotheses

The following hypotheses can be tested. The difference in χ^2 for H_1 vs. H_2, or H_3 vs. H_4 provides a test of the hypothesis that $\rho = 1$.

$$\begin{aligned} H_1 &: \rho = 1 \text{ and } H_2 \\ H_2 &: \begin{cases} \beta_1 = \beta_2 & \theta_1^2 = \theta_2^2 \\ \beta_3 = \beta_4 & \theta_3^2 = \theta_4^2 \end{cases} \\ H_3 &: \rho = 1, \text{ all other parameters free} \\ H_4 &: \text{all parameters free} \end{aligned}$$

H_1 and H_2 assume the measures x_1, x_2 and y_1, y_2 are parallel. H_3 and H_4 assume they are merely congeneric.

Several Sets of Congeneric Tests

These four hypotheses actually form a 2×2 factorial

- parallel vs. congeneric: H_1 and H_2 vs. H_3 and H_4 and
- $\rho = 1$ vs. $\rho \neq 1$.

For nested models, model comparisons can be done by testing the difference in χ^2, or by comparing other fit statistics (AIC, BIC, RMSEA, etc.)

- LISREL can fit multiple models, but you have to do the model comparison tests "by hand."
- AMOS can fit multiple models, and does the model comparisons for you.
- With PROC CALIS, the CALISCMP macro provides a flexible summary of multiple-model comparisons.

Example: Lord's data

Lord's vocabulary test data:

- x_1, x_2: two 15-item tests, liberal time limits
- y_1, y_2: two 75-item tests, highly speeded

Analyses of these data give the following results:

Hypothesis	Free Parameters	df	χ^2	p-value	AIC
H_1: par, $\rho = 1$	4	6	37.33	0.00	25.34
H_2: par	5	5	1.93	0.86	-8.07
H_3: cong, $\rho = 1$	8	2	36.21	0.00	32.27
H_4: cong	9	1	0.70	0.70	-1.30

- Models H2 and H4 are acceptable, by χ^2 tests
- Model H2 is "best" by AIC

Lord's data

The tests of $\rho = 1$ can be obtained by taking the differences in χ^2,

	Parallel χ^2	df	Congeneric χ^2	df
$\rho = 1$	37.33	6	36.21	2
$\rho \neq 1$	1.93	5	0.70	1
	35.40	1	35.51	1

- Both tests reject the hypothesis that $\rho = 1$,
- Under model H2, the ML estimate is $\widehat{\rho} = 0.889$.
- $\Rightarrow$ speeded and unspeeded vocab. tests do not measure *exactly* the same thing.
  SAS example: datavis.ca/courses/factor/sas/calis1c.sas

Lord's data: PROC CALIS

```
data lord(type=cov);
   input _type_ $ _name_ $  x1 x2 y1 y2;
datalines;
n     .  649     649     649     649
cov   x1 86.3937  .       .       .
cov   x2 57.7751 86.2632  .       .
cov   y1 56.8651 59.3177 97.2850  .
cov   y2 58.8986 59.6683 73.8201 97.8192
mean  .    0       0       0       0
;
```

Model H4: $\beta_1, \beta_2, \beta_3, \beta_4 \ldots \rho$=free

```
title "Lord's data: H4- unconstrained two-factor model";
proc calis data=lord
      cov
      summary   outram=M4;
   lineqs  x1   = beta1 F1   + e1,
           x2   = beta2 F1   + e2,
           y1   = beta3 F2   + e3,
           y2   = beta4 F2   + e4;
   std  F1 F2 = 1 1,
        e1 e2 e3 e4 = ve1 ve2 ve3 ve4;
   cov  F1 F2 = rho;
run;
```


Lord's data: PROC CALIS

The SUMMARY output contains many fit indices:

```
      Lord's data: H4- unconstrained two-factor model

 Covariance Structure Analysis: Maximum Likelihood Estimation

 Fit criterion . . . . . . . . . . . . . . . . . .      0.0011
 Goodness of Fit Index (GFI) . . . . . . . . . . .      0.9995
 GFI Adjusted for Degrees of Freedom (AGFI)  . . .      0.9946
 Root Mean Square Residual (RMR) . . . . . . . . .      0.2715
 Chi-square = 0.7033         df = 1      Prob>chi**2 = 0.4017
 Null Model Chi-square:      df = 6                 1466.5884
 Bentler's Comparative Fit Index . . . . . . . . .      1.0000
 Normal Theory Reweighted LS Chi-square  . . . . .      0.7028
 Akaike's Information Criterion  . . . . . . . . .     -1.2967
 Consistent Information Criterion  . . . . . . . .     -6.7722
 Schwarz's Bayesian Criterion  . . . . . . . . . .     -5.7722
 McDonald's (1989) Centrality. . . . . . . . . . .      1.0002
 Bentler & Bonett's (1980) Non-normed Index. . . .      1.0012
 Bentler & Bonett's (1980) Normed Index. . . . . .      0.9995
 James, Mulaik, & Brett (1982) Parsimonious Index.      0.1666
 ...
```


Lord's data: PROC CALIS

Model H3: H4, with $\rho = 1$

```
title "Lord's data: H3- rho=1, one-congeneric factor";
proc calis data=lord
     cov summary outram=M3;
   lineqs  x1  = beta1 F1 + e1,
           x2  = beta2 F1 + e2,
           y1  = beta3 F2 + e3,
           y2  = beta4 F2 + e4;
   std  F1 F2 = 1 1,
        e1 e2 e3 e4 = ve1 ve2 ve3 ve4;
   cov  F1 F2 = 1;
run;
```

Model H2: $\beta_1 = \beta_2, \beta_3 = \beta_4$ ..., ρ=free

```
title "Lord's data: H2- X1 and X2 parallel, Y1 and Y2 parallel";
proc calis data=lord
     cov summary outram=M2;
   lineqs  x1  = betax F1  + e1,
           x2  = betax F1  + e2,
           y1  = betay F2  + e3,
           y2  = betay F2  + e4;
   std  F1 F2 = 1 1,
        e1 e2 e3 e4 = vex vex vey vey;
   cov  F1 F2 = rho;
run;
```

Model comparisons using CALISCMP macro and the OUTRAM= data sets

```
%caliscmp(ram=M1 M2 M3 M4,
   models=%str(H1 par rho=1/H2 par/H3 con rho=1/H4 con),
   compare=1 2 / 3 4 /1 3/ 2 4);
```

Model Comparison Statistics from 4 RAM data sets

Model	Parameters	df	Chi-Square	P>ChiSq	RMS Residual	GFI	AIC
H1 par rho=1	4	6	37.3412	0.00000	2.53409	0.97048	25.3412
H2 par	5	5	1.9320	0.85847	0.69829	0.99849	-8.0680
H3 con rho=1	8	2	36.2723	0.00000	2.43656	0.97122	32.2723
H4 con	9	1	0.7033	0.40168	0.27150	0.99946	-1.2967

(more fit statistics are compared than shown here.)


```
%caliscmp(ram=M1 M2 M3 M4,
   models=%str(H1 par rho=1/H2 par/H3 con rho=1/H4 con),
   compare=1 2 / 3 4 /1 3/ 2 4);
```

Model Comparison Statistics from 4 RAM data sets

Model Comparison	ChiSq	df	p-value	
H1 par rho=1 vs. H2 par	35.4092	1	0.00000	****
H3 con rho=1 vs. H4 con	35.5690	1	0.00000	****
H1 par rho=1 vs. H3 con rho=1	1.0689	4	0.89918	
H2 par vs. H4 con	1.2287	4	0.87335	

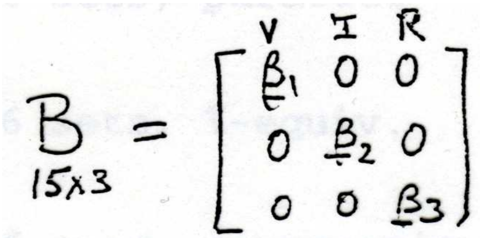
If the measures are cross-classified in two or more ways, it is possible to test equivalence at the level of each way of classification.
Lord (1956) examined the correlations among 15 tests of three types:

- Vocabulary, Figural Intersections, and Arithmetic Reasoning.
- Each test given in two versions: Unspeeded (liberal time limits) and Speeded.

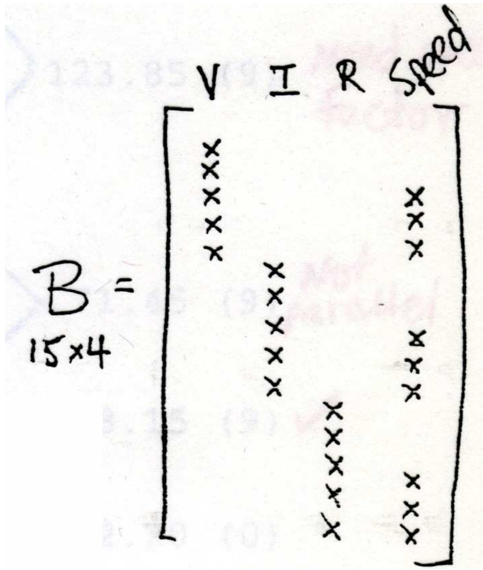
The goal was to identify factors of performance on speeded tests:

- Is speed on cognitive tests a unitary trait?
- If there are several type of speed factors, how are they correlated?
- How highly correlated are speed and power factors on the same test?

Hypothesized factor patterns (B):



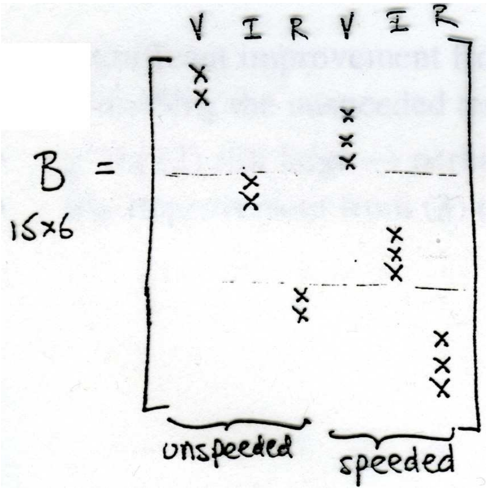
(1) 3 congeneric sets



(2) 3 congeneric sets + speed factor

Example: Speeded and Non-speeded tests

Hypothesized factor patterns (B):



(3)-(6) Six factors

- (3) parallel: equal β & θ^2 for each factor
- (4) τ-equivalent: equal β in each col
- (5) congeneric: no equality constraints
- (6) six factors: 3 content, 3 speed

Results:

Hypothesis	Parameters	df	χ^2	$\Delta\chi^2$ (df)
1: 3 congeneric sets	33	87	264.35	
2: 3 sets + speed factor	42	78	140.50	123.85 (9)
3: 6 sets, parallel	27	93	210.10	
4: 6 sets, τ-equiv.	36	84	138.72	71.45 (9)
5: 6 sets, congeneric	45	75	120.57	18.15 (9)
6: 6 factors	45	75	108.37	12.20 (0)

Notes:

- Significant improvement from (1) to (2) $\rightarrow$ speeded tests measure something the unspeeded tests do not.
- χ^2 for (2) still large $\rightarrow$ perhaps there are different kinds of speed factors.
- Big improvement from (3) to (4) $\rightarrow$ not parallel

Outline

Factorial Invariance

Multi-sample analyses:

- When a set of measures have been obtained from samples from several populations, we often wish to study the similarities in factor structure across groups.
- The ACOVS/LISREL model allows any parameter to be assigned an arbitrary fixed value, or constrained to be equal to some other parameter. We can test any degree of invariance from totally separate factor structures to completely invariant ones.
- Model
  Let $\boldsymbol{x}_g$ be the vector of tests administered to group $g, g = 1, 2, \ldots, m$, and assume that a factor analysis model holds in each population with some number of common factors, k_g.

$$\Sigma_g = \Lambda_g \Phi_g \Lambda_g^{\mathsf{T}} + \Psi_g$$

Factorial Invariance: Examples

- Arguably among the most important recent development in personality psychology is the idea that individual differences in personality characteristics is organized into five main trait domains: Extraversion, Agreeableness, Conscientiousness, Neuroticism, and Openness
  - One widely used instrument is the 60-item NEO-Five factor inventory (Costa & McCrae, 1992), developed and analyzed for a North American, English-speaking population
  - To what extent does the same factor structure apply across gender?
  - To what extent does the same factor structure applies in other cultural and language goups?
- The emerging field of cross-cultural psychology offers many similar examples.

Factorial Invariance: Hypotheses

We can examine a number of different hypotheses about how "similar" the covariance structure is across groups.

Hypotheses

- Can we simply pool the data over groups?
- If not, can we say that the same number of factors apply in all groups?
- If so, are the factor loadings equal over groups?
- What about factor correlations and unique variances?

Software

- LISREL, AMOS, and M Plus all provide convenient ways to do multi-sample analysis.
- `PROC CALIS` in SAS 9.3 does too.
- In R, the `lavaan` package package provides multi-sample analysis and the `measurementInvariance()` function. The `sem` package package includes a `multigroupModel()` for such models

1. Equality of Covariance Matrices

$$H_{=\Sigma} : \Sigma_1 = \Sigma_2 = \cdots = \Sigma_m$$

If this hypothesis is tenable, there is no need to analyse each group separately or test further for differences among them: Simply pool all the data, and do one analysis!
If we reject $H_{=\Sigma}$, we may wish to test a less restrictive hypothesis that posits some form of invariance.
The test statistic for $H_{=\Sigma}$ is

$$\chi^2_{=\Sigma} = n \log |S| - \sum_{g=1}^{m} n_g \log |S_g|$$

which is distributed approx. as χ^2 with $d_{=\Sigma} = (m-1)p(p-1)/2$ df.
(This test can be carried out in SAS with `PROC DISCRIM` using the `POOL=TEST` option)

2. Same number of factors (Configural invariance)
The least restrictive form of "invariance" is simply that the number of factors is the same in each population:

$$H_k : k_1 = k_2 = \cdots = k_m = \text{ a specified value, } k$$

This can be tested by doing an unrestricted factor analysis for k factors on each group separately, and summing the χ^2's and degrees of freedom,

$$\chi^2_k = \sum_g^m \chi^2_k(g) \qquad d_k = m \times [(p-k)^2 - (p+k)]/2$$

3. Same factor pattern (Weak invariance)
If the hypothesis of a common number of factors is tenable, one may proceed to test the hypothesis of an invariant factor pattern:

$$H_\Lambda : \Lambda_1 = \Lambda_2 = \cdots = \Lambda_m$$

The common factor pattern Λ may be either completely unspecified, or be specified to have zeros in certain positions.

To obtain a χ^2 for this hypothesis, estimate Λ (common to all groups), plus $\Phi_1, \Phi_2, \ldots, \Phi_m$, and $\Psi_1, \Psi_2, \ldots, \Psi_m$, yielding a minimum value of the function, F. Then, $\chi^2_\Lambda = 2 \times F_{min}$.
To test the hypothesis H_Λ, *given that the number of factors is the same* in all groups, use

$$\chi^2_{\Lambda|k} = \chi^2_\Lambda - \chi^2_k \text{ with } d_{\Lambda|k} = d_\Lambda - d_k \text{ degrees of freedom}$$

4. Same factor pattern and unique variances (Strong invariance)
A stronger hypothesis is that the unique variances, as well as the factor pattern, are invariant across groups:

$$H_{\Lambda\Psi} : \begin{cases} \Lambda_1 = \Lambda_2 = \cdots = \Lambda_m \\ \Psi_1 = \Psi_2 = \cdots = \Psi_m \end{cases}$$

5. Same factor pattern, means and unique variances (Strict invariance) The strongest hypothesis is that the factor means are also equal across groups as well as the factor patterns and unique variances:

$$H_{\Lambda\Psi\mu} : \begin{cases} \Lambda_1 = \Lambda_2 = \cdots = \Lambda_m \\ \Psi_1 = \Psi_2 = \cdots = \Psi_m \\ \mu_1 = \mu_2 = \cdots = \mu_m \end{cases}$$

Example: Academic and Non-Academic Boys

Sorbom (1976) analyzed STEP tests of reading and writing given in grade 5 and grade 7 to samples of boys in Academic and Non-Academic programs.

Data

	Academic ($N = 373$)				Non-Acad ($N = 249$)			
Read Gr5	281.35				174.48			
Writ Gr5	184.22	182.82			134.47	161.87		
Read Gr7	216.74	171.70	283.29		129.84	118.84	228.45	
Writ Gr7	198.38	153.20	208.84	246.07	102.19	97.77	136.06	180.46

Hypotheses

The following hypotheses were tested:

Hypothesis	Model specifications
A. $H_{=\Sigma} : \Sigma_1 = \Sigma_2$	$\begin{cases} \mathbf{\Lambda}_1 = \mathbf{\Lambda}_2 = \mathbf{I}_{(4\times4)} \\ \mathbf{\Psi}_1 = \mathbf{\Psi}_2 = \mathbf{0}_{(4\times4)} \\ \mathbf{\Phi}_1 = \mathbf{\Phi}_2 \text{ constrained, free} \end{cases}$
B. $H_{k=2} : \Sigma_1, \Sigma_2$ both fit with $k = 2$ correlated factors	$\begin{cases} \mathbf{\Lambda}_1 = \mathbf{\Lambda}_2 = \begin{bmatrix} x & 0 \\ x & 0 \\ 0 & x \\ 0 & x \end{bmatrix} \\ \mathbf{\Phi}_1, \mathbf{\Phi}_2, \mathbf{\Psi}_1, \mathbf{\Psi}_2 \text{ free} \end{cases}$
C. $H_\Lambda : H_{k=2}$ & $\mathbf{\Lambda}_1 = \mathbf{\Lambda}_2$	$\mathbf{\Lambda}_1 = \mathbf{\Lambda}_2$ (constrained)
D. $H_{\Lambda,\Theta} : H_\Lambda$ & $\mathbf{\Psi}_1 = \mathbf{\Psi}_2$	$\begin{cases} \mathbf{\Psi}_1 = \mathbf{\Psi}_2 \text{ (constrained)} \\ \mathbf{\Lambda}_1 = \mathbf{\Lambda}_2 \end{cases}$
E. $H_{\Lambda,\Theta,\Phi} : H_{\Lambda,\Theta}$ & $\mathbf{\Phi}_1 = \mathbf{\Phi}_2$	$\begin{cases} \mathbf{\Phi}_1 = \mathbf{\Phi}_2 \text{ (constrained)} \\ \mathbf{\Psi}_1 = \mathbf{\Psi}_2 \\ \mathbf{\Lambda}_1 = \mathbf{\Lambda}_2 \end{cases}$

Analysis

The analysis was carried out with both LISREL and AMOS. AMOS is particularly convenient for multi-sample analysis, and for testing a series of nested hypotheses.

Summary of Hypothesis Tests for Factorial Invariance

Hypothesis	Overall fit				Group A		Group N-A	
	χ^2	df	prob	AIC	GFI	RMSR	GFI	RMSR
A: $H_{=\Sigma}$	38.08	10	.000	55.10	.982	28.17	.958	42.26
B: $H_{k=2}$	1.52	2	.468	37.52	.999	0.73	.999	0.78
C: H_Λ	8.77	4	.067	40.65	.996	5.17	.989	7.83
D: $H_{\Lambda,\Psi}$	21.55	8	.006	44.55	.990	7.33	.975	11.06
E: $H_{\Lambda,\Psi,\Phi}$	38.22	11	.000	53.36	.981	28.18	.958	42.26

- The hypothesis of equal factor loadings (H_Λ) in both samples is tenable.
- Unique variances appear to differ in the two samples.
- The factor correlation (ϕ_{12}) appears to be greater in the Academic sample than in the non-Academic sample.

Factorial Invariance: LISREL syntax

Model B for Academic group: 2 correlated, non-overlapping factors

```
Ex12: 2 Correlated factors: Hypothesis B (Group A)
DAta NGroup=2 NI=4 NObs=373
LAbels  file=lisrel12.dat; CMatrix file=lisrel12.dat
MOdel NX=4 NKSI=2
! Pattern: 1 free parameter and 1 fixed parameter in each column
FRee    LX(2,1) LX(4,2)
STart 1 LX(1,1) LX(3,2)
OUtput
```

Model B for Non-Academic group: same pattern as Group A

```
Ex12: 2 Correlated factors: Hypothesis B (Group N-A)
DAta NObs=249
LAbels  file=lisrel12.dat; CMatrix file=lisrel12.dat
MOdel LX=PS
PDiagram
OUtput
```

`LX=PS`: same pattern and starting values as in previous group but loadings are not constrained to be equal

Factorial Invariance: LISREL syntax

Model C for Academic group: equal Λ_x— same as Model B

```
Ex12: Equal Lambda: Hypothesis C (Group A)
DAta NGroup=2 NI=4 NObs=373
LAbels  file=lisrel12.dat; CMatrix file=lisrel12.dat
MOdel NX=4 NKSI=2
! Pattern: 1 free parameter and 1 fixed parameter in each column
FRee    LX(2,1) LX(4,2)
STart 1 LX(1,1) LX(3,2)
OUtput
```

Model C for Non-Academic group: same Λ_x as Group A

```
Ex12: 2 Correlated factors: Hypothesis B (Group N-A)
DAta NObs=249
LAbels  file=lisrel12.dat; CMatrix file=lisrel12.dat
MOdel LX=INvariant
PDiagram
OUtput
```

`LX=IN`: loadings constrained to be equal to those in Group A
Complete example:
datavis.ca/courses/factor/lisrel/lisrel12.ls8

Factorial Invariance: AMOS Basic syntax

Model B for Academic group:

```
Sub Main
  Dim Sem As New AmosEngine
  With Sem
   .title "Academic and NonAcademic Boys (Sorbom, 1976): "  _
        & "Equality of Factor Structures"

  ' Start out with the least constrained model
     .Model "B: 2 Factors, unconstrained"
     .BeginGroup "invar.xls", "Academic"
     .GroupName "Academic Boys"
     .Structure "Read_Gr5 = ( 1 ) Gr5 + (1) eps1"
     .Structure "Writ_Gr5 = (L1a) Gr5 + (1) eps2"
     .Structure "Read_Gr7 = ( 1 ) Gr7 + (1) eps3"
     .Structure "Writ_Gr7 = (L2a) Gr7 + (1) eps4"
     .Structure "Gr5 <--> Gr7 (phi1)"
     .Structure "eps1 (v1a)"
     .Structure "eps2 (v2a)"
     .Structure "eps3 (v3a)"
     .Structure "eps4 (v4a)"
```


Factorial Invariance: AMOS Basic syntax

Model B for Non-Academic group:

```
     .BeginGroup "invar.xls", "NonAcademic"
     .GroupName "NonAcademic Boys"
     .Structure "Read_Gr5 = ( 1 ) Gr5 + (1) eps1"
     .Structure "Writ_Gr5 = (L1b) Gr5 + (1) eps2"
     .Structure "Read_Gr7 = ( 1 ) Gr7 + (1) eps3"
     .Structure "Writ_Gr7 = (L2b) Gr7 + (1) eps4"
     .Structure "Gr5 <--> Gr7 (phi2)"
     .Structure "eps1 (v1b)"
     .Structure "eps2 (v2b)"
     .Structure "eps3 (v3b)"
     .Structure "eps4 (v4b)"
```

Note that the model is the same, but all parameter names are suffixed with 'b', so they are not constrained to be equal

Factorial Invariance: AMOS Basic syntax

Now, other models can be specified in terms of equality constraints across groups:

```
  ' Fix the loadings in the two groups to be equal
     .Model "C: = loadings", _
      "L1a = L1b; L2a = L2b"

  ' Add contraint that unique variances are equal
     .Model "D: C + = unique var", _
      "L1a=L1b; L2a=L2b; _
       v1a=v1b; v2a=v2b;  v3a=v3b;  v4a=v4b"

  ' Add contraint that factor correlations are equal
     .Model "E: D + = factor corr", _
      "L1a=L1b; L2a=L2b; _
       v1a=v1b; v2a=v2b;  v3a=v3b;  v4a=v4b; _
       phi1=phi2"
   End With
End Sub
```


lavaan package: Factorial invariance tests

Data

Data for Academic and Non-academic boys:

```
library(sem)
Sorbom.acad <- read.moments(diag=TRUE,
names=c('Read.Gr5', 'Writ.Gr5', 'Read.Gr7', 'Writ.Gr7'))
281.349
184.219 182.821
216.739 171.699 283.289
198.376 153.201 208.837 246.069

Sorbom.nonacad <- read.moments(diag=TRUE,
names=c('Read.Gr5', 'Writ.Gr5', 'Read.Gr7', 'Writ.Gr7'))
174.485
134.468 161.869
129.840 118.836 228.449
102.194  97.767 136.058 180.460

# make the two matrices into a list
Sorbom <- list(acad=Sorbom.acad, nonacad=Sorbom.nonacad)
```


lavaan package: Factorial invariance tests I

Model

Specify `lavaan` model for 2 correlated, non-overlapping factors:

```
library(lavaan)
Sorbom.model <-
'G5 =~ Read.Gr5 + Writ.Gr5
 G7 =~ Read.Gr7 + Writ.Gr7 '
```

Run a cfa model (testing k=2 for each group):

```
(Sorbom.cfa <- cfa(Sorbom.model, sample.cov=Sorbom, sample.nobs=c(373,249))

Lavaan (0.4-7) converged normally after 240 iterations
  Number of observations per group
  acad                                                373
  nonacad                                             249

  Estimator                                            ML
  Minimum Function Chi-square                       1.525
  Degrees of freedom                                    2
  P-value                                           0.467

Chi-square for each group:
  acad                                              0.863
  nonacad                                           0.662
```


Tests of measurement invariance I

Test all models of measurement invariance:

```
measurementInvariance(Sorbom.model, sample.cov=Sorbom,
    sample.nobs=c(373,249))

Measurement invariance tests:

Model 1: configural invariance:
    chisq        df    pvalue       cfi     rmsea       bic
    1.525     2.000     0.467     1.000     0.000 18788.554

Model 2: weak invariance (equal loadings):
    chisq        df    pvalue       cfi     rmsea       bic
    8.806     4.000     0.066     0.997     0.062 18782.970

[Model 1 versus model 2]
  delta.chisq      delta.df delta.p.value      delta.cfi
        7.282         2.000         0.026          0.003
```

Tests of measurement invariance II

```
Model 3: strong invariance (equal loadings + intercepts):
    chisq         df     pvalue        cfi      rmsea        bic
    8.806      6.000      0.185      0.998      0.039 18821.567

[Model 1 versus model 3]
  delta.chisq       delta.df delta.p.value      delta.cfi
        7.282          4.000         0.122          0.002

[Model 2 versus model 3]
  delta.chisq       delta.df delta.p.value      delta.cfi
        0.000          2.000         1.000         -0.001
 ...
```

A fourth model also tests equality of means, but means are not available for this example.

Outline

Identifiability in CFA models

- Because they offer the possibility of fitting hypotheses that are partially specified, care must be take with CFA models to ensure that a unique solution can be found.
- For an unrestricted factor model with k latent factors, we have seen that at least k^2 restrictions *must* be imposed.
- It turns out that this is a ***necessary***, but ***not a sufficient*** condition for the model to be identified.

Identifiability in CFA models

Identifiability

- In addition, it is necessary to specify the *unit of measurement*, or *scale* for each latent variable. This may be done by (arbitrarily) assigning one fixed non-zero loading, typically 1, in each column of the factor matrix.
- For a problem with 6 variables and 2 factors, the loading matrix would look like this:

$$\Lambda_2 = \begin{bmatrix} 1 & 0 \\ x & x \\ x & x \\ 0 & 1 \\ x & x \\ x & x \end{bmatrix} \quad \Phi_2 = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$$

The fixed 1s correspond to equating the scale of factor 1 to that of variable 1, and factor 2 is equated to variable 4.

Identifiability condition

- Let $\boldsymbol{\theta}$ be the $t \times 1$ vector containing all unknown parameters in the model, and let
$$\boldsymbol{\Sigma}(\boldsymbol{\theta}) = \boldsymbol{\Lambda}\boldsymbol{\Phi}\boldsymbol{\Lambda}^{\mathsf{T}} + \boldsymbol{\Psi}$$
be the covariance matrix expressed in terms of those parameters.
- Then, the parameters in $\boldsymbol{\theta}$ are identified if you can show that the elements of $\boldsymbol{\theta}$ are uniquely determined functions of the elements of $\boldsymbol{\Sigma}$, that is:
$$\boldsymbol{\Sigma}(\boldsymbol{\theta}_1) = \boldsymbol{\Sigma}(\boldsymbol{\theta}_2) \rightarrow \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$$
- (It is not always easy to show this, particularly for complex models)
- Sign of lack of identification: model does not converge.

Identifiability condition

For example, for a 1-factor, 3-variable model:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} [\xi] + \begin{bmatrix} z_1 & & \\ & z_2 & \\ & & z_3 \end{bmatrix}$$

Then, letting $\boldsymbol{\Phi} = var(\xi) = 1$ and $var(z_i) = \psi_i$, the covariance matrix of the observed variables can be expressed as:

$$\boldsymbol{\Sigma}(\boldsymbol{\theta}) = \begin{bmatrix} \lambda_1^2 + \psi_1 & & \\ \lambda_2\lambda_1 & \lambda_2^2 + \psi_2 & \\ \lambda_3\lambda_1 & \lambda_3\lambda_1 & \lambda_3^2 + \psi_3 \end{bmatrix}$$

Each parameter can be solved for, so the model is identified.

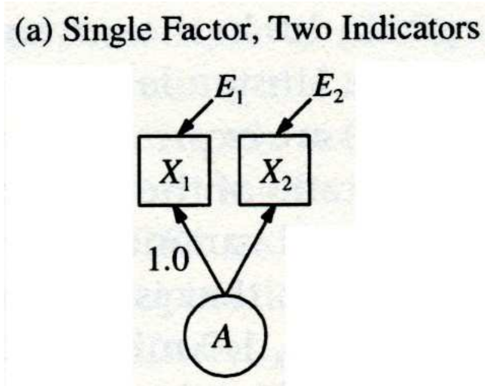
Identifiability rules: t rule

t-rule: There cannot be more unknown parameters than there are known elements in the sample covariance matrix. This is a ***necessary***, but ***not sufficient*** condition.

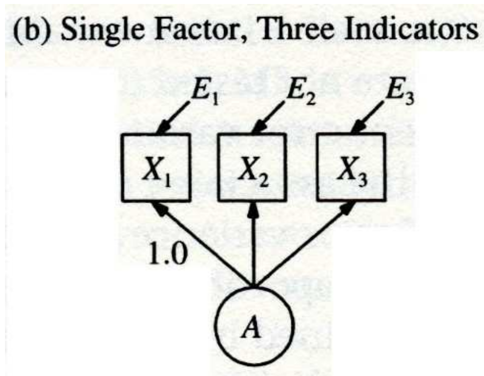
$$t \leq p(p+1)/2$$

Example:
For 6 tests, you can estimate no more than $6 \times 7/2 = 21$ parameters.

Identifiability rules: t rule



2 variables: 3 var-covariances
model: 4 free parameters
not identified



3 variables: 6 var-covariances
model: 6 free parameters
just identified

Identifiability rules: 3 variable rules

3-variable rules: A factor model is identified when there are:

- three or more variables per factor
- each variable has one and only one non-zero loading
- the unique factors are uncorrelated ($\boldsymbol{\Psi}$ is a diagonal matrix).

There are no restrictions on the factor correlations ($\boldsymbol{\Phi}$). These conditions are jointly ***sufficient***, though ***not necessary***.

Identifiability rules: 2 variable rules

2-variable rules A less restrictive set of rules is:

- two or more variables per factor
- each variable has one and only one non-zero loading
- each factor is scaled by setting one $\lambda_{ij} = 1$ in each column.
- the unique factors are uncorrelated ($\boldsymbol{\Psi}$ is a diagonal matrix).
- there are no fixed zero elements in $\boldsymbol{\Phi}$.

These conditions are also ***sufficient***, though ***not necessary***.
Example:
With 4 variables, and 2 latent variables, the model is identified if the parameters are specified as

$$\Lambda = \begin{bmatrix} 1 & 0 \\ \lambda_{21} & 0 \\ 0 & 1 \\ 0 & \lambda_{42} \end{bmatrix}, \qquad \Phi = \begin{bmatrix} \phi_{11} & \\ \phi_{21} & \phi_{22} \end{bmatrix} = \text{free}$$

Power and Sample Size for EFA and CFA

Bad news Determining the required sample size, or the power of a hypothesis test are far more complex in EFA and CFA than in other statistical applications (correlation, ANOVA, etc.)

Good news There are a few things you *can* do to choose a sample size intelligently.

Power and Sample Size for EFA and CFA

Rules of thumb for EFA

For EFA, there is little statistical basis for determining the appropriate sample size, and little basis for determining power (but the overall approach of CFA can be used).
Some traditional "rules of thumb" for EFA:

- ***The more the better!***
  - Reliability and replicability increase directly with $\sqrt{N}$.
  - More reliable factors can be extracted with larger sample sizes.
- Absolute minimum– $N = 5p$, but you should have $N > 100$ for any non-trivial factor analysis. Minimum applies only when communalities are high and p/k is high.
- Most EFA and CFA studies use $N > 200$, some as high as 500-600.
- Safer to use at least $N > 10p$.
- The lower the reliabilities, the larger N should be.

Using desired standard errors

- An alternative approach for EFA considers the standard errors of correlations, in relation to sample size.
- This usually provides more informed guidance than the rules of thumb. It can be shown that,

$$\sigma(\rho) = \frac{1-\rho^2}{\sqrt{N}} + \mathcal{O}(N^{-1})$$

so, we could determine the sample size to make the standard error of a "typical" correlation smaller than some given value.

$$\sqrt{N} > \frac{1-\rho^2}{\sigma(\rho)}$$

Using desired standard errors

	Sample size				
ρ	50	100	200	400	800
0.1	0.140	0.099	0.070	0.050	0.035
0.3	0.129	0.091	0.064	0.046	0.032
0.5	0.106	0.075	0.053	0.038	0.027
0.7	0.072	0.051	0.036	0.026	0.018

- Standard error decreases as $|\rho|$ increases.
- So, if you want to keep the standard error less than 0.05, you need $N = 400$ when the "typical" correlation is only 0.1, but $N = 100$ when the "typical" correlation is 0.7.
- In many behavioural and psychology studies, correlations among *different scales* are modest, at best ($0.1 \le \rho \le 0.3$).
- For typical scale analysis, one should expect the correlations among items on the *same scale* to be much higher ($0.7 \le \rho \le 0.9$), $\Rightarrow$ smaller required sample size for the same standard error.

Power and Sample size for CFA

- ***Problems:*** The situation in CFA wrt power analysis is typically reversed compared with other forms of hypothesis tests—
  - $\chi^2 = (N-1)F_{min}$, so large $N \Rightarrow$ reject H_0.
  - With small specification errors, large sample size will magnify their effects $\Rightarrow$ reject H_0.
  - With large specification errors, small sample size will mask their effects $\Rightarrow$ accept H_0.
- ***Overall approach:*** MacCallum, Browne and Sugawara (1996) approach allows for testing a null hypothesis of 'not-good-fit', so that a significant result provides support for good fit.
  - Effect size is defined in terms of a null hypothesis and alternative hypothesis value of the root-mean-square error of approximation (RMSEA) index. Typical values for RMSEA:

$\le .05$	close fit
$.05 - .08$	fair
$.08 - .10$	mediocre
$> .10$	poor

  - These values, together with the df for the model being fitted, sample size (N), and error rate (α), allow power to be calculated.

Power and Sample size for CFA

- The `CSMPOWER` macro
  - See: `http://datavis.ca/sasmac/csmpower.html`
  - Retrospective power analysis— uses the RMSEA values from the OUTRAM= data set from PROC CALIS for the model fitted.
  - Prospective power analysis— values of RMSEA, DF and N must be provided through the macro arguments.

Example: Retrospective power analysis

Here, we examine the power for the test of Lord's two-factor model for speeded and unspeeded vocabulary tests, where $N = 649$.

```
title "Power analysis: Lord's Vocabulary Data";
title2 "Lord's data: H1- X1 and X2 parallel,
        Y1 and Y2 parallel, rho=1";
proc calis data=lord cov summary outram=ram1;
   lineqs  x1  = betax F1  + e1,
           x2  = betax F1  + e2,
           y1  = betay F2  + e3,
           y2  = betay F2  + e4;
   std  F1 F2 = 1 1,
        e1 e2 e3 e4 = vex vex vey vey;
   cov  F1 F2 = 1;
run;

*-- Power analysis from RMSEA statistics in this model;
title 'Retrospective power analysis';
%csmpower(data=ram1);
```


Example: Retrospective power analysis

Results include:

Alpha	df	Name of Variable	N	H0 fit value	Ha fit value	Power
0.05	6	RMSEAEST	649	0.05	0.08977	0.75385
		RMSEALOB	649	0.05	0.06349	0.19282
		RMSEAUPB	649	0.05	0.11839	0.99202

With this sample size, we have power of 0.75 to distinguish between a fit with RMSEA=0.05 and one with RMSEA=0.09.

Example: Prospective power analysis

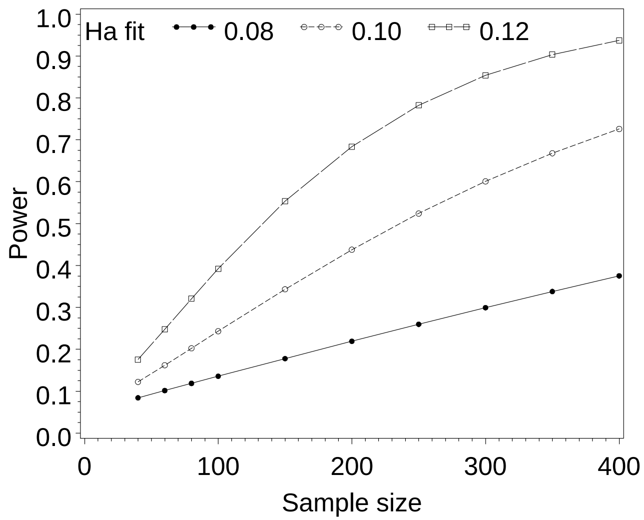
For prospective power analysis, we specify the RMSEA for alternative hypotheses of 'not good fit' with the RMSEAA= parameter (for H_a).

```
*--; title  'Prospective power analysis';
%csmpower(df=6, rmseaa=%str(.08 to .12 by .02),
          plot=%str(power*n =rmseaa));
```

Results include a listing:

Alpha	df	N	H0 fit value	Ha fit value	Power
0.05	6	40	0.05	0.08	0.08438
		40	0.05	0.10	0.12243
		40	0.05	0.12	0.17575
		60	0.05	0.08	0.10168
		60	0.05	0.10	0.16214
		60	0.05	0.12	0.24802
		80	0.05	0.08	0.11883
		80	0.05	0.10	0.20262
		80	0.05	0.12	0.32093
		100	0.05	0.08	0.13585
		100	0.05	0.10	0.24333
		100	0.05	0.12	0.39214
		...	...	...	...
		400	0.05	0.08	0.37545
		400	0.05	0.10	0.72599
		400	0.05	0.12	0.93738

Plot of Power by N for each level of RMSEAA:



- For the most stringent test of H_0 : RMSEA $= 0.05$ vs. H_a : RMSEA $= 0.08$, the largest sample size, $N = 400$ only provides a power of 0.375.
- Good thing they used $N = 649$!

Individual model specifications

- The overall approach can only evaluate power or required sample size for the whole model.
- It does not distinguish among the *a priori* specifications of free and fixed parameters implied by the model being tested.

Things become more difficult when the focus is on power for deciding on some one or a few specifications (parameters) in a model.

Individual model specifications

There are some promising results:

- Satorra (1989) found that the modification indices ("Lagrange multipliers" in `PROC CALIS`)— $\Delta\chi^2$ for *fixed parameters* in a model approximate the χ^2 non-centrality parameters required to determine power for a specific fixed parameter.
- Similarly, the Wald tests, $\chi_1^2 = (\text{parm}/s(\text{parm}))^2$ approximate the χ^2 non-centrality parameters required to determine power for *free parameters*.
- These χ^2 values should be studied in relation to the estimated change in the parameter (ECP).
  - A large $\Delta\chi^2$ with a small ECP simply reflects the high power to detect small differences which comes with large *N*.
  - Similarly, a small $\Delta\chi^2$ with a large ECP reflects low power for large differences with small *N*.

  See the paper by Kaplan, "Statistical power in structural equation models", `www.gsu.edu/~mkteer/power.html` for further discussion and references on these issues.

Summary: Part 3

- **Confirmatory Factor Analysis**
  - Specify a model by imposing restrictions on free parameters (fixed or constrained)
  - Fit model by minimizing $F(\boldsymbol{S}, \hat{\Sigma})$ w.r.t. free parameters
  - $(N-1)F_{min} \sim \chi^2$ gives goodness-of-test of the model as specified
- **More general models**
  - 2nd order factor models: model structure of 1st order factors
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