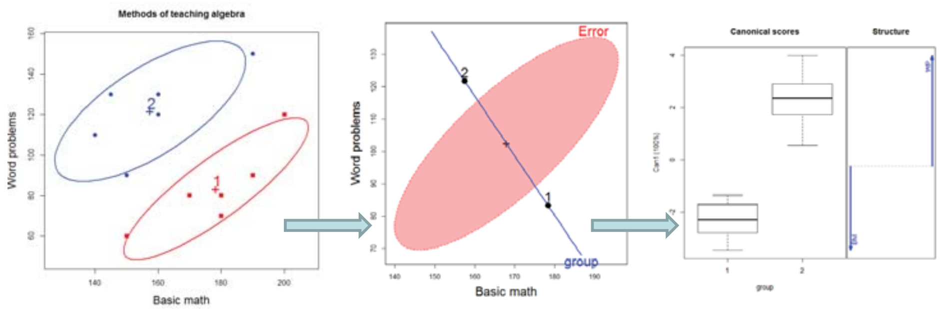

Hotelling's T² & MANOVA

Testing & understanding multivariate mean differences

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Psychology 6140

Tests of mean differences

- Why study more than 1 criterion?
  - More complete description of the phenomenon under investigation
    - Job satisfaction: work load, peer relations, autonomy
    - Student achievement: math, science, literature, french
    - Therapy outcome: self-worth, relationship, health
  - Treatment may affect subjects in more than one way
    - All up/down together?
    - Some up, some down?
  - These effects may be correlated
  - Same arguments as for multivariate regression.

Tests of mean differences

- Why use a multivariate analysis (T^2, MANOVA)?
  - Separate univariate tests → inflated error rate (α)
    e.g., 10 responses, separate ANOVAs at α=.05
    $\Pr(\text{at least 1 error}) \geq 1-(.95)^{10} \approx 0.40 \leq 10\,\alpha$
  - Multivariate tests: give an overall test at α=.05
  - Univariate tests ignore correlations among the responses
  - Groups may not differ on any single criterion, but may differ on **several criteria jointly**

Simpler methods don't get things quite right

TABLE 4.2
Experimentwise Error Rates for Analyzing Multivariate Data with only Univariate Tests and with a Multivariate Test followed by Univariate Tests*

much too liberal

somewhat conservative

Sample Size	*Number of Variables*	*Proportion of variance in common* .10	.30	.50	.70
	Univariate Tests Only				
10	3	.145	.112	.114	.077
10	6	.267	.190	.178	.111
10	9	.348	.247	.209	.129
30	3	.115	.119	.117	.085
30	6	.225	.200	.176	.115
30	9	.296	.263	.223	.140
50	3	.138	.124	.102	.083
50	6	.230	.190	.160	.115
50	9	.324	.258	.208	.146
	Multivariate Test Followed by Univariate Tests				
10	3	.044	.029	.035	.022
10	6	.046	.029	.030	.017
10	9	.050	.026	.025	.018
30	3	.037	.044	.029	.025
30	6	.037	.037	.032	.021
30	9	.042	.042	.030	.021
50	3	.038	.041	.033	.028
50	6	.037	.039	.028	.027
50	9	.036	.038	.026	.020

*Nominal α = .05.

Tests of mean differences

- Why not *always* use MANOVA tests?
  - Like overall *F*-test in ANOVA---
    - Many responses, small effects → lower power for the overall test on all responses.
    - As usual, hypothesis tests should be focused on research questions, rather than “shotgun” approach
  - Somewhat harder to describe / explain
    - Multiple test statistics: Wilks’, HLT, Pillai, Roy
    - But: all are equivalent for s=1 tests, so no problem
    - just report the equivalent F (all the same for T^2 here)

Hotelling’s T^2

- Multivariate generalization of univariate *t*-test
  - *t*-test :: ANOVA as T^2 :: MANOVA
- T^2 as maximum univariate t^2 for a linear combination of responses
- Special case of the GLH: **L B M** = **0**
- Introduces ideas of discriminant analysis
- Introduces ideas of MANOVA

In the history of multivariate statistical methods, there are a few important ideas that paved the way from univariate → bivariate → multivariate. This is one of them.

T^2: generalized *t* test

- 1 sample, univariate
- $H_0 : \mu = \mu_0$
- $H_1 : \mu \neq \mu_0$

$$t = \frac{(\bar{x} - \mu_0)\sqrt{N}}{s}$$

$$t^2 = \frac{N(\bar{x} - \mu_0)^2}{s^2} = N(\bar{x} - \mu_0)(s^2)^{-1}(\bar{x} - \mu_0)$$

- 1 sample, multivariate
- $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_{0\,(p \times 1)}$
- $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$

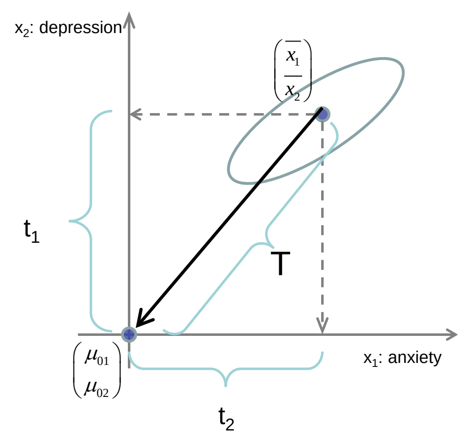
$$T^2 = \underbrace{N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)\mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)}$$

Squared distance between mean vector and hypothesis, in the metric of S^{-1}

Thus, T^2 is like the square of a univariate *t* statistic

T^2: generalized *t* test

Does exam period increase anxiety *and* depression?



$$T^2 = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)\mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$$
$$= N\, D^2(\bar{\mathbf{x}}, \boldsymbol{\mu}_0)$$

Squared distance between mean vector and hypothesis, in the metric of S^{-1}

Q: For the *same* means, would T^2 be larger or smaller if $r < 0$?

T^2: max possible t^2 for linear combination

- For data matrix $\mathbf{X}_{(n \times p)}$, consider linear comb with weights $\mathbf{a}_{(p \times 1)}$:

$$\mathbf{w} = \mathbf{X}\,\mathbf{a}$$

- Then $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_{0\,(p \times 1)} \rightarrow H_0 : \mu_w = \mathbf{a}'\boldsymbol{\mu}_0$
- Find weights to give max t^2:

$$t(\mathbf{a}) = \frac{\bar{w} - \mu_w}{\sqrt{s_w^2 / N}} = \frac{\mathbf{a}'(\bar{s} - \mu_0)\sqrt{N}}{\sqrt{\mathbf{a}'\mathbf{S}\mathbf{a}}} \longrightarrow \begin{array}{c} [N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)' - \lambda \mathbf{S}]\mathbf{a} = 0 \\ \text{or,} \quad (\mathbf{Q}_H - \lambda \mathbf{Q}_E)\mathbf{a} = 0 \end{array}$$

Rank ($\mathbf{Q}_H$)=1 → 1 non-zero latent root

T^2: max possible t^2 for linear combination

- Hence, the maximum value of $t^2(\mathbf{a})$ is

$$\lambda = N(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)'\mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0) = T^2$$

(the 1 non-zero latent root)

- The corresponding latent vector, **a**, is

$$\mathbf{a} = \mathbf{S}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu}_0)$$

(raw) discriminant function coefficients. These give the relative contribution of each response to T^2

- In the two-sample case, analogous results using $(\bar{\mathbf{x}}_1 - \bar{\mathbf{x}}_2)$

T^2: critical values

- For a single response, $t^2_{N-1} \sim F(1, N-1)$
- Since we chose **a** to give max $t^2(\mathbf{a})$, need to take this into account.
- Hotelling showed that a transformation of T^2 has an *F* distribution

$$F^* = \frac{N-p}{p(N-1)} T^2 \quad \sim \quad F(p, N-p)$$

- In SAS/SPSS/R, we typically use equivalent F values based on Roy's test or HLT

T^2: invariance

- Univariate *t* unchanged under any *linear* transformation

$$x \rightarrow a\,x + b$$

- T^2 is invariant under all *affine* transformations

$$\mathbf{x}_{(p \times 1)} \rightarrow \mathbf{C}_{(p \times p)}\,\mathbf{x} + \mathbf{d} \quad : \mathbf{C} \text{ non-singular}$$

- The same is true for all MANOVA tests

T^2: confidence regions for means

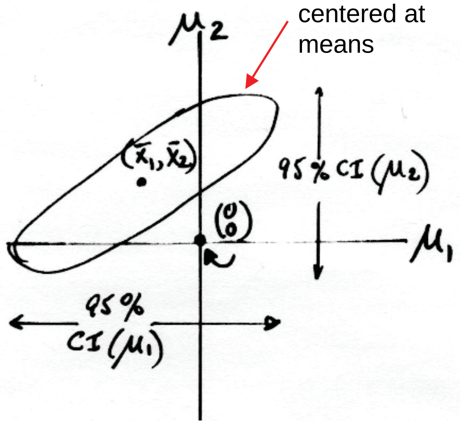
A 1-α confidence region for **μ** (p x 1) consists of those **μ** for which

$$N(\bar{x}-\mu)' S^{-1}(\bar{x}-\mu) \le \frac{p(N-1)}{N-p} F_{1-\alpha}(p, N-p)$$

Convert F back to T^2

If $\underset{\sim}{\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is outside the region, $H_0: \underset{\sim}{\mu} = \underset{\sim}{0}$ is rejected.

Data ellipse centered at means



In the figure, note that H0: **μ** = **0** is rejected, but the **univariate tests are not rejected**.

T^2: special case of GLM

The multivariate General Linear Model expresses all models in the form

$$\mathbf{Y}_{n\times p} = \mathbf{X}_{n\times q}\, \mathbf{B}_{q\times p} + \mathbf{E}_{n\times p}$$

A two-sample T2 takes the form:

$$\begin{array}{c} G_1 \\ G_2 \end{array} \begin{bmatrix} Y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{\mu}' \\ \underset{\sim}{\alpha}_1' \\ \underset{\sim}{\alpha}_2' \end{bmatrix} + \begin{bmatrix} E \end{bmatrix}$$

The usual LS estimates:

$$\hat{\mathbf{B}} = \begin{pmatrix} \hat{\boldsymbol{\mu}}' \\ \widehat{\boldsymbol{\alpha}_1}' \\ \widehat{\boldsymbol{\alpha}_2}' \end{pmatrix} = (\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Y}$$

T^2: special case of GLH

- All hypotheses are of the form

$$H_0 : \ \mathbf{L}\ \mathbf{B}\ \mathbf{M} = \mathbf{0}$$

Contrasts among groups

Contrasts among variables (profile analysis)

- Eg H_0: $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 \rightarrow \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2 = 0$

$$\mathbf{LBM} = (0 \quad 1 \quad -1) \begin{pmatrix} \boldsymbol{\mu}' \\ \boldsymbol{\alpha}_1' \\ \boldsymbol{\alpha}_2' \end{pmatrix} \times \mathbf{I}_{3\times 3} = \mathbf{0}$$

Example: methods of teaching algebra

- Responses: students evaluated on--
  - Basic math (BM)
  - Word problems (WP)
- Groups: two instructors, different teaching methods (presumably: equal ability, random assignment)

```
title 'Two-Sample Hotelling T2';
data mathscor;
   input group BM WP;
   label BM='Basic Math'
         WP='Word Problems';
datalines;
1 190  90
1 170  80
1 180  80
1 200 120
1 150  60
1 180  70
2 160 120
2 190 150
2 150  90
2 160 130
2 140 110
2 145 130
;
```

Univariate tests

	t	*p*-value
BM	2.06	0.066
WP	-3.23	0.009**


```
proc glm data=mathscor outstat=HEstats;
   class group;
   model BM WP = group / nouni;
   manova H=group;
run;
```

```
        MANOVA Test Criteria and Exact F Statistics for
           the Hypothesis of No Overall group Effect
                H = Type III SSCP Matrix for group
                     E = Error SSCP Matrix

                        S=1    M=0    N=3.5

Statistic                        Value    F Value    Num DF    Den DF    Pr > F

Wilks' Lambda               0.13481891      28.88         2         9    0.0001
Pillai's Trace              0.86518109      28.88         2         9    0.0001
Hotelling-Lawley Trace      6.41735719      28.88         2         9    0.0001
Roy's Greatest Root         6.41735719      28.88         2         9    0.0001
```

$\eta^2 = 1 - \Lambda = 0.865$ $T^2 = df_e \times \lambda_1 = (n_1 + n_2 - 2) \times HLT = 10\ HLT = 64.17$

All F values are equal when $s = df_h = 1$

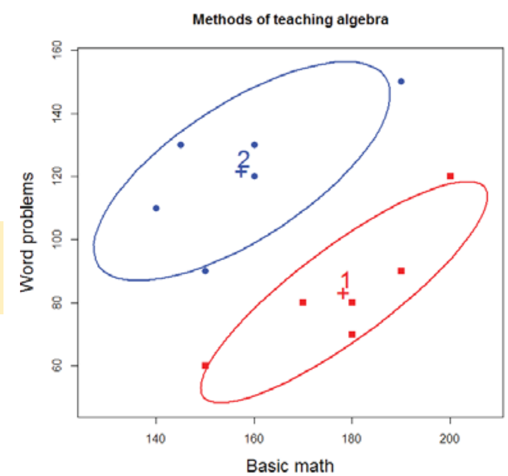
Analysis with R:

```
> mod <- lm(cbind(BM, WP) ~ group, data=mathscore)
> Anova(mod)

Type II MANOVA Tests: Pillai test statistic
      Df test stat approx F num Df den Df    Pr(>F)
group  1   0.86518   28.878      2      9 0.0001213 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
>
> print(summary(Anova(mod)), SSP=FALSE)

Type II MANOVA Tests:
------------------------------------------

Term: group
Multivariate Tests: group
                 Df test stat approx F num Df den Df    Pr(>F)
Pillai            1    0.8652   28.878      2      9 0.0001213 ***
Wilks             1    0.1348   28.878      2      9 0.0001213 ***
Hotelling-Lawley  1    6.4174   28.878      2      9 0.0001213 ***
Roy               1    6.4174   28.878      2      9 0.0001213 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```


Visualizing T^2 & MANOVA

Data ellipses show:
- group means (centroids)
- within-group scatter
- (are they ~ the same?)

```
covEllipses(mathscore[,2:3],
  mathscore$group,
  pooled=FALSE)
```



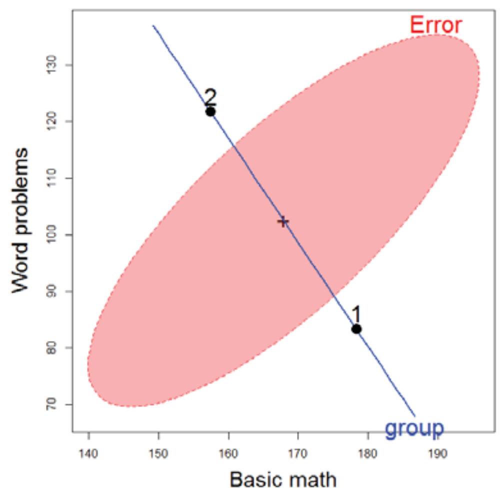
Interpretation: Instructor 1 focuses more on Basic Math; instructor 2 on Word Problems

This bivariate display is much more informative than separate univariate displays for BM and WP!

Visualizing T^2 & MANOVA

HE plots show:
- variation of means (**H**)
- pooled within-group scatter (**E**)

```
heplot(mod, fill=TRUE,
  xlab="Basic math",
  ylab="Word problems")
```

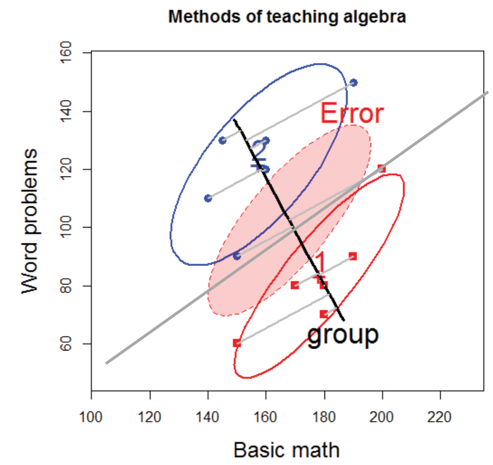


Visualizing T^2 & MANOVA

Overlaying these shows:
- variation of means (**H**)
- pooled within-group scatter (**E**) = average of within-group var-cov matrices
- discriminant axis = **H**

Discriminant analysis:

- Find linear comb of BM, WP to best discriminate between groups
- DA also allows different *n* s of groups and cost of mis-classification (shifts boundary line)



Assumptions: homogeneity of (co)variance

- For univariate *t*-test or ANOVA, we assume equal variance within groups
  - $s^2_1 = s^2_2 = \ldots = s^2_g \rightarrow s^2_{\text{pooled}}$ or MSE
  - Box test or Levine's test often used
- Multivariate tests: translates to equality of within-group covariance matrices,
  - $S_1 = S_2 = \ldots = S_g \rightarrow S_{\text{pooled}}$ = E matrix
  - Box M test: $H_0 : \Sigma_1 = \Sigma_2 = \ldots = \Sigma_g$

$$M = \frac{\prod |S_i|^{N_i/2}}{|S|^{N/2}_{pooled}} \quad \rightarrow \chi 2 \text{ with } (g-1)p(p+1)/2 \text{ df}$$

  - SAS: proc discrim, `pool=test` option
  - Why: discriminant function is linear of $S_1 = S_2 = \ldots$ but quadratic otherwise
  - R: **heplots::boxM()**


```
proc discrim data=mathscor
     pool=test
     canonical ncan=1 out=scores;
   class group;
   var BM WP;
```

```
                The DISCRIM Procedure
   Test of Homogeneity of Within Covariance Matrices

      Chi-Square        DF      Pr > ChiSq       ✓
       0.617821         3         0.8923
```

Raw Canonical Coefficients

Variable	Can1
BM	-0.0835
WP	0.0753

group	BM	WP	Can1
1	190	90	-2.78495
1	170	80	-1.86756
1	180	80	-2.70261
1	200	120	-1.36188
1	150	60	-1.70287
1	180	70	-3.45531
2	160	120	1.97832
2	190	150	1.73129
2	150	90	0.55525
2	160	130	2.73102
2	140	110	2.89571
2	145	130	3.98360

(I'm using this as a short cut to also introduce canonical analysis)

$T^2 = t^2$ based on Can1

```
proc ttest data=scores;
   class group;
   var can1;
run;
```

A simple *t* test using canonical scores is equivalent to Hotelling's T^2

T-Tests

Variable	Method	Variances	DF	t Value	Pr > |t|
Can1	Pooled	Equal	10	-8.01	<.0001
Can1	Satterthwaite	Unequal	8.8	-8.01	<.0001

$T^2 = (-8.01)^2 = 64.17$

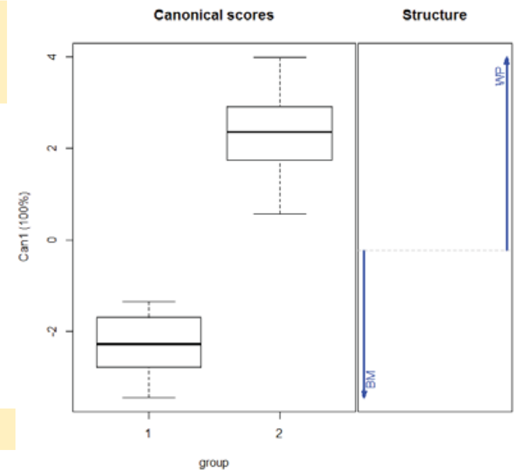
Canonical analysis in R with the candisc package

```
library(candisc)
mod.can <- candisc(mod)
summary(mod.can)
```

Canonical Discriminant Analysis for group:

```
  CanRsq Eigenvalue Difference Percent Cumulative
1 0.8652      6.417                100        100

Class means:
[1] -2.313  2.313

std coefficients:
    BM     WP
-1.463  1.546
```

```
plot(mod.can)
```



boxM(mod)

Box's M-test for Homogeneity of Covariance Matrices

Chi-Sq (approx.) = 0.618, df = 3, p-value = 0.89

Example: Delay in oral practice

- Postovsky (1970): Two groups were studied in second language learning (*n*=28 each)
  - Control: 0 delay before starting oral practice
  - Expt'l: 4 week delay before starting oral practice
  - NB: groups were matched on age, education, former language training & language aptitude [Why?]
- Responses (exam after 6 weeks):
  - listen
  - speak
  - read
  - write


```
data oral;
    input subjno grp listen speak read write;
    length Group $ 8;
    if grp=1 then Group='Exptl';
    if grp=2 then Group='Control';
datalines;
 1  1  34  66  39  97
 2  1  35  60  39  95
 3  1  32  57  39  94
 4  1  29  53  39  97
 5  1  37  58  40  96
 6  1  35  57  34  90
 7  1  34  51  37  84
 8  1  25  42  37  80
  ...
```

Complete output on class web: SAS examples/glm/oral.sas

```
proc glm data=oral outstat=HEstats;
     class group;
     model listen--write = group;
     manova h = group / short;
run;
```

MANOVA Test Criteria and Exact F Statistics for
the Hypothesis of No Overall Group Effect
H = Type III SSCP Matrix for Group
E = Error SSCP Matrix

S=1 M=1 N=24.5

Statistic	Value	F Value	Num DF	Den DF	Pr > F
Wilks' Lambda	0.914414	1.19	4	51	0.3250
Pillai's Trace	0.085586	1.19	4	51	0.3250
Hotelling-Lawley	0.093596	1.19	4	51	0.3250
Roy's Greatest Root	0.093596	1.19	4	51	0.3250

$\eta^2 = 1 - \Lambda = 0.086$

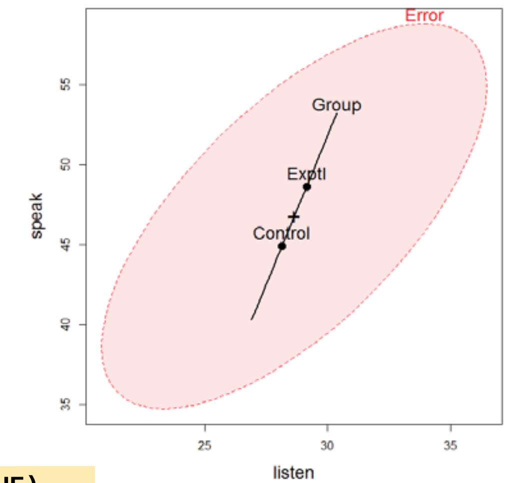
NB: $s = \min(p, df_h)$, where $df_h = g-1$

```
oral.mod <- lm(cbind(listen, speak, read, write) ~ Group, data=oral)
Anova(mod)
```


Visualizing the analysis

HE plot shows that:

- Exptl > Control on both measures
- Diff^{ce} too small, relative to error variation, to be considered significant
- Residuals are also positively correlated
- The H "ellipse" becomes a line when s=1

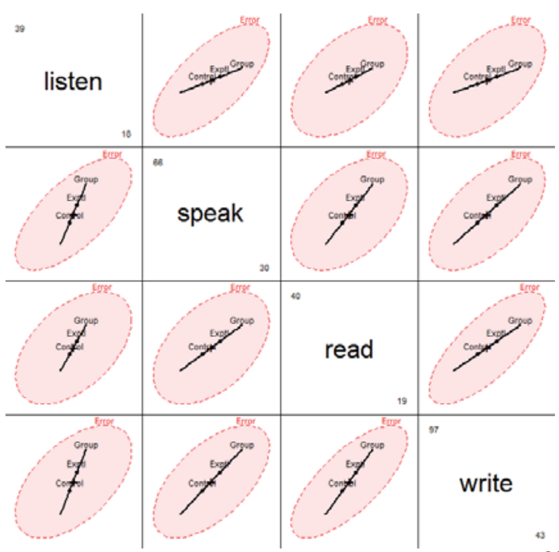



```
heplot(oral.mod, fill=TRUE)
```



```
pairs(oral.mod, fill=TRUE, ...)
```

HE plot matrix confirms that this is true for all pairs of responses



Profile analysis (repeated measures)

- When the responses are commensurate (on the same scale) it is often useful to test equal means across responses
- Univariate analysis (repeated measures) and multivariate analysis (profile analysis) frame similar questions, in different ways
  - Parallel profiles: No Group x Measure interaction
  - Equal levels: No Group main effect
  - Flatness: No Measure main effect
- Univariate analysis relies on additional assumptions (compound symmetry), not always met in data

Example: one-sample profile analysis

- Study of immediate memory in a sentence probe task
  - Ex: *The tall man met the young girl who got the new hat*

    Pos: 1 2 3 4 5

    Function: Adj_1 Subj Adj_2 Obj Rel.PN
  - IV: sentence position
  - DV: response speed (~ 1/RT)
- Goals:
  - Test equal means across sentence position
  - Test contrasts among means based on sentence structure
    - Subject phrase vs. predicate phrase 1 1 -1 -1 0
    - Adj vs. Noun 1 -1 1 -1 0
    - Interaction: (subj-predicate) x (adj-noun) 1 -1 -1 1 0
    - main vs. relative clause 1 1 1 1 -4


```
data probe1;
   input subjno P1-P5;
datalines;
 1 51 36 50 35 42
 2 27 20 26 17 27
 3 37 22 41 37 30
 4 42 36 32 34 27
 5 27 18 33 14 29
 6 43 32 43 35 40
 7 41 22 36 25 38
 8 38 21 31 20 16
 9 36 23 27 25 28
10 26 31 31 32 36
11 29 20 25 26 25
;
```

```
proc transpose data=probe1
      out=long;
  by subjno;
  var p1-p5;
```

The probe1 dataset is read here in 'wide' format, suitable for

- Multivariate analysis
- Repeated measures analysis

For plotting, traditional univariate and mixed-model analysis, it must be in 'long' format

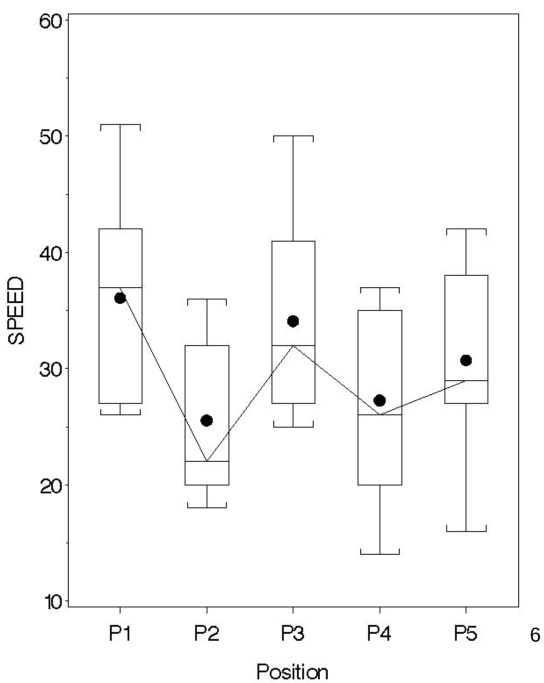
subjno	Position	Speed
1	P1	51
1	P2	36
1	P3	50
1	P4	35
1	P5	42
2	P1	27
2	P2	20
2	P3	26
2	P4	17
2	P5	27
…		

For plotting, use the data in the long format

```
%boxplot(data=long,
     var=Speed,
     class=Position,
     connect=1);
```

Note:

- measured response: RT
- speed ~ 1 / RT
- is there still a skewness problem?



Profile analysis: GLH

- H_0: equal means over positions (flatness):

  $\mu_1 = \mu_2 = \ldots = \mu_5$
- GLH, using **L**=1 and

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix} \Rightarrow H_0 : \begin{bmatrix} \mu_1 - \mu_5 \\ \mu_2 - \mu_5 \\ \mu_3 - \mu_5 \\ \mu_4 - \mu_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

  - These are "simple contrasts" against a baseline level
  - For overall test, any **M** with rank=4 and cols that sum = 0 gives same result
  - Equivalent to analysis of **Y M**

```
proc glm data=probe1;
  model P1-P5 = / nouni;
  manova H= intercept ;          /* No position effect*/
```

```
            MANOVA Test Criteria and Exact F Statistics
              for the Hypothesis of no POSITION Effect
                H = Type III SSCP Matrix for POSITION
                        E = Error SSCP Matrix

                         S=1     M=1     N=2.5

Statistic                Value    F Value    Num DF    Den DF    Pr > F

Wilks' Lambda         0.248225       5.30         4         7    0.0277
Pillai's Trace        0.751775       5.30         4         7    0.0277
Hotelling-Lawley      3.028595       5.30         4         7    0.0277
Roy's Greatest Root   3.028595       5.30         4         7    0.0277
```

Complete output on class web: SAS examples/glm/probe1.sas

In R:

```
probe1.mod <- lm(cbind(p1, p2, p3, p4, p5) ~ 1, data=probe1)
idata <- data.frame(position=factor(1:5))
Anova(probe1.mod, idata=idata, idesign = ~ position)
```


The same hypothesis can be tested using the `repeated` statement:

```
proc glm data=probe1;
   model P1-P5 = / nouni;
   repeated Position 5 contrast(5);
```

In addition to the multivariate test, we get the traditional univariate tests. However, these rely on additional assumptions (more later).

Repeated Measures Analysis of Variance
Univariate Tests of Hypotheses for Within Subject Effects

Source	DF	SS	Mean Sq	F Value	Pr > F	Adj Pr > F G – G	Adj Pr > F H - F
POSITION	4	867.527	216.881	9.25	<.0001	0.0001	<.0001
Error(POSITION)	40	938.072	23.452				

Greenhouse-Geisser Epsilon 0.7851
Huynh-Feldt Epsilon 1.1860

Profile analysis: GLH contrasts

- Here, we also want to test specific contrasts among the sentence positions

$$\mathbf{M} = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix} \end{array} \begin{array}{l} Adj_1 \\ Subj \\ Adj_2 \\ Obj \\ \mathrm{Re}\,l \end{array}$$

- In proc glm, any custom M can be defined in the `manova` statement


```
proc glm data=probe1;
   model P1-P5 = / nouni;
   manova H= intercept              /* No position effect    */
          M= P1 + P2 - P3 - P4,     /* Subject vs. Predicate */
             P1 - P2 + P3 - P4,     /* Adjs vs Nouns         */
             P1 - P2 - P3 + P4,     /* SubPred x AdjNoun     */
             P1 + P2 + P3 + P4 - 4*P5 /* Relative clause     */
          mnames = SubjPred AdjNoun SPxAN RelPN
          / summary printH printE SHORT ;
```

M Matrix Describing Transformed Variables

	P1	P2	P3	P4	P5
SUBPRED	1	1	-1	-1	0
ADJNOUN	1	-1	1	-1	0
SPxAN	1	-1	-1	1	0
RELPN	1	1	1	1	-4

H & E matrices

H is labeled 'Intercept' since H_0 is mean=0

Diag elements are univariate SS for each contrast

H = Type III SSCP Matrix for Intercept

	SubjPred	AdjNoun	SPxAN	RelPN
SubjPred	0.82	52.09	11.18	0.27
AdjNoun	52.09	3316.45	711.91	17.36
SPxAN	11.18	711.91	152.82	3.73
RelPN	0.27	17.36	3.73	0.09

E = Error SSCP Matrix

	SUBPRED	ADJNOUN	SPxAN	RELPN
SUBPRED	694.18	54.91	23.82	562.72
ADJNOUN	54.91	1370.55	-48.91	53.64
SPxAN	23.82	-48.91	482.18	863.27
RELPN	562.72	53.64	863.27	6026.91

Multivariate Analysis of Variance

Dependent Variable: SUBPRED

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Intercept	1	0.81818	0.8181818	0.01	0.9157
Error	10	694.18182	69.4181818		

Dependent Variable: ADJNOUN

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Intercept	1	3316.4545	3316.454545	24.20	0.0006
Error	10	1370.5456	137.054545		

Dependent Variable: SPxAN

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Intercept	1	152.81818	152.8181818	3.17	0.1054
Error	10	482.18182	48.2181818		

Dependent Variable: RELPN

Source	DF	Type III SS	Mean Square	F Value	Pr > F
Intercept	1	0.0909	0.090909	0.00	0.9904
Error	10	6026.9091	602.690909		

So, these are based on the diag elements of H and E

Two-sample profile analysis

- Same probe task, but with two groups:
  - Gp 1: low STM capacity
  - Gp 2: high STM capacity
- Questions:
  - Are profiles parallel? (Group x Position)
  - Equal levels? (Group main effect)
  - Flat profiles? (Position main effect)

Two-sample profile analysis: GLH

- Parallelism:

Interaction of Group x Position:

are successive differences the same for Gp 1 & 2?

$$H_{01}: \begin{bmatrix} \mu_{11} - \mu_{12} \\ \mu_{12} - \mu_{13} \\ \mu_{13} - \mu_{14} \\ \mu_{14} - \mu_{15} \end{bmatrix} = \begin{bmatrix} \mu_{21} - \mu_{22} \\ \mu_{22} - \mu_{23} \\ \mu_{23} - \mu_{24} \\ \mu_{24} - \mu_{25} \end{bmatrix}$$

- **L** = (1 -1)

contrast for groups

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

profile contrasts for positions

Two-sample profile analysis: GLH

- Equal levels (Group effect)
  - H_0: $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 \rightarrow \mathbf{1}'\boldsymbol{\mu}_1 - \mathbf{1}'\boldsymbol{\mu}_2 = 0$ (in univariate tests)
  - GLH:

$$\mathbf{L} = (1 \;\; -1) \qquad \mathbf{M} = \mathbf{I}_{(5x5)}$$

- Flatness (Position effect)
  - H0: $(\mu_{11}+\mu_{21}) = \ldots = (\mu_{15}+\mu_{25})$
  - GLH:

$$\mathbf{L} = (1 \;\; 1) \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$


```
proc glm data=probe2;
   class group;
   model p1-p5 = group / nouni;
   repeated position 5 profile / short;
   manova h = group / printe printh short;
   title2 'Two Sample Profile Analysis'; run;
```

repeated measures tests (univariate)

multivariate tests

Position effect:

Statistic	Value	F Value	Num DF	Den DF	Pr > F
Wilks' Lambda	0.21986	13.31	4	15	<.0001
Pillai's Trace	0.78013	13.31	4	15	<.0001
Hotelling-Lawley	3.54825	13.31	4	15	<.0001
Roy's Greatest Root	3.54825	13.31	4	15	<.0001

Position x Group effect:

Statistic	Value	F Value	Num DF	Den DF	Pr > F
Wilks' Lambda	0.83906	0.72	4	15	0.5919
Pillai's Trace	0.16094	0.72	4	15	0.5919
Hotelling-Lawley	0.19181	0.72	4	15	0.5919
Roy's Greatest Root	0.19181	0.72	4	15	0.5919

Group effect: H_0: $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$

Statistic	Value	F Value	Num DF	Den DF	Pr > F
Wilks' Lambda	0.55608	2.24	5	14	0.1083
Pillai's Trace	0.44392	2.24	5	14	0.1083
Hotelling-Lawley	0.79832	2.24	5	14	0.1083
Roy's Greatest Root	0.79832	2.24	5	14	0.1083

Compare with univariate, repeated measures approach: $\mathbf{1}'\boldsymbol{\mu}_1 - \mathbf{1}'\boldsymbol{\mu}_2 = 0$

Repeated Measures Analysis of Variance
Tests of Hypotheses for Between Subjects Effects

Source	DF	SS	Mean Square	F Value	Pr > F
group	1	1772.41	1772.4100	8.90	0.0080
Error	18	3583.14	199.0633		

The univariate test looks at only one contrast among the many tested by the multivariate test.

Univariate, repeated measures tests rely on further assumption:
- Compound symmetry (= correlations among repeated measures)
- Univariate adjustments (G-G, H-F) adjust the p-values to take violations into account

Repeated Measures Analysis of Variance
Univariate Tests of Hypotheses for Within Subject Effects

Source	DF	SS	MS	F Value	Pr > F	Adj Pr > F G - G	Adj Pr > F H - F
position	4	3371.30	842.825	14.48	<.0001	<.0001	<.0001
position*group	4	79.94	19.985	0.34	0.8479	0.8068	0.8479
Error(position)	72	4191.96	58.222				

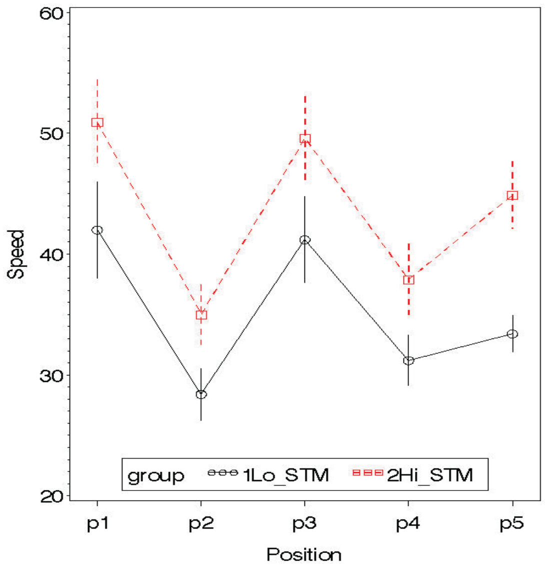
Greenhouse-Geisser Epsilon 0.8009
Huynh-Feldt Epsilon 1.0487

Complete output on class web: SAS examples/glm/probe2.sas

Plotting means: %meanplot

Reshape data to long format:

```
proc transpose data=probe2
               out=long;
   var p1-p5;
   by group subjno;
   run;
data long;
   set long;
   rename col1=Speed
         _name_=Pos;
   label _name_='Position';
   run;

%meanplot(data=long,
    var=Speed,
    class=Pos Group);
```

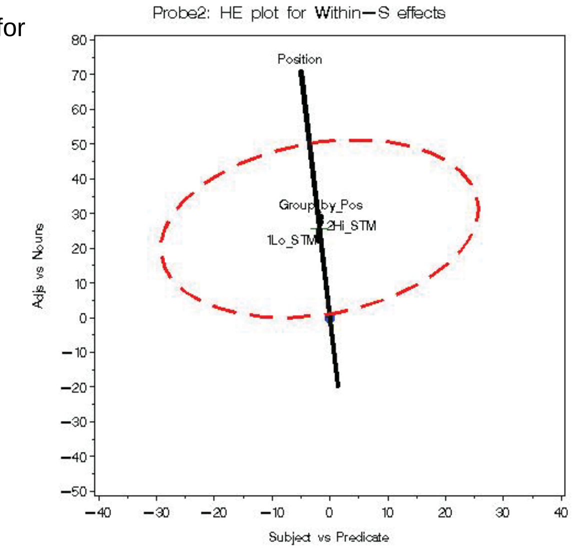


HE plots

Using substantive contrasts for the sentence positions:

$$\mathbf{M} = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix} \end{array} \begin{array}{l} Adj_1 \\ Subj \\ Adj_2 \\ Obj \\ Rel \end{array}$$

HE plot shows the position effect is mainly in the Adj vs Noun contrast



Summary

- Hotelling's T^2 introduces general ideas:
  - T^2: multivariate analog of t^2
  - Special case of GLH for 1- & 2-sample design
  - $T^2 \sim \lambda_1$ (Roy test), eigenvector → discriminant weights
  - Multiple groups: *t*-test :: ANOVA as T^2 :: MANOVA
  - Specific tests: contrasts among groups (**L**) and among responses (**M**) – better than all pairwise!
  - Visualizing hypothesis and error variation via HE plots
    - 1 df tests: H "ellipse" is a line
    - Orientation: Shows variation of means wrt responses